Multivariable Calculus

Math 212 Fall 2005 — Fowler 111 MWF 9:30pm - 10:25am

 — http://faculty.oxy.edu/ron/math/212/05/

Class 1: **Friday September 2**

SUMMARY Introduction to Vectors and Notation
CURRENT READING Williamson & Trotter, Section 1.1
HOMEWORK Williamson & Trotter, §1.1 # 2, 3, 7, 8, 9, 22, 25; §1.2 # 1, 4, 7, 8, 21, 22

Notation

One of the main concepts of the class will be thinking of multi-dimensional space. You will recall that in two-dimensions (i.e. $\mathbb{R}^2$) we denote points in space using ordered pairs (x, y) called co-ordinates. In three-dimensions ($\mathbb{R}^3$) this becomes the ordered triple (x, y, z). Points in n-dimensional space ($\mathbb{R}^n$) are denoted by $(x_1, x_2, x_3, \ldots, x_n)$ called n-tuples.

Vectors and Scalars

We shall refer to these pairs, triples and n-tuples as **vectors**. The book will use a bold letter $\mathbf{x}$ to denote a vector, but I will generally use the notation $\vec{x}$. In addition to vectors, we also have objects called **scalars** which are numerical quantities like mass or temperature which are measured on a numerical scale as opposed to vector quantities such as velocity or force which have *both* magnitude and direction.

Scalar Multiplication and Vector Addition

Consider the scalar $r = 2$ and vectors $\vec{a} = (1, 2)$ and $\vec{b} = (-3, 4)$.

$r\vec{a} =$ ______________________ $\vec{a}+\vec{b} =$ ______________________

Vectors can actually be defined more generally and theoretically as ojects which belong to a **vector space**, which is defined below. For most of Math 212, we will just be thinking of a vector as an element of $\mathbb{R}^n$.

DEFINITION

A **vector space** is a set $\mathcal{V}$ of objects (of *any* kind!) called **vectors**, with two operations, called **vector addition** and **scalar multiplication**, that satisfy the following ten properties:

(1) $\mathcal{V}$ is closed under vector addition: $\vec{v}, \vec{w} \in \mathcal{V} \Rightarrow (\vec{v} + \vec{w}) \in \mathcal{V}$
(2) $\mathcal{V}$ is closed under scalar multiplication: $c \in \mathbb{R}, \vec{v} \in \mathcal{V} \Rightarrow (c\vec{v}) \in \mathcal{V}$
(3) Vector addition is commutative: $\vec{v} + \vec{w} = \vec{w} + \vec{v}$
(4) Vector addition is associative: $(\vec{v} + \vec{w}) + \vec{z} = \vec{v} + (\vec{w} + \vec{z})$
(5) There is a unique additive identity in $\mathcal{V}$: $\vec{v} + \vec{0} = \vec{v}$
(6) Each element in $\mathcal{V}$ has a unique additive inverse: $\vec{v} + (-\vec{v}) = \vec{0}$
(7) The scalar 1 acts as the multiplicative identity: $1\vec{v} = \vec{v}$
(8) Scalar multiplication is associative: $c_1(c_2\vec{v}) = (c_1c_2)\vec{v}$.
(9) Scalar multiplication is distributive w.r.t. vector addition: $c(\vec{v} + \vec{w}) = c\vec{v} + c\vec{w}$.
(10) Scalar multiplication is distributive w.r.t. scalar addition. $(c_1 + c_2)\vec{v} = c_1\vec{v} + c_2\vec{v}$.

DEFINITION

A space is said to be "closed under an operation" when the result of the operation applied to any element of the space, is also an element of the space.

GROUPWORK

Prove that $\mathbb{R}^2$ is a vector space.

Distance and Length

The **distance** between two points in $\mathbb{R}^n$ is the **length** of the vector representing the difference between the position vectors for the two points.

The **length** of a vector $\vec{x}$ in $\mathbb{R}^n$ is denoted by $|\vec{x}|$ and computed by $\sqrt{x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2}$
Thus, given a point $\vec{p}$ and another point $\vec{q}$ the distance between them is $|\vec{p} - \vec{q}|$ (or, equivalently $|\vec{q} - \vec{p}|$)

Examples

Find the length of $(0, 4, -1, 3)$ and the distance between $(-1, 2, 1)$ and $(3, 0, 0)$.

EXERCISE

Find the midpoint between the points $(-1, 2, 5)$ and $(2, 0, 4)$.

Arrow

The **directed line segment** a.k.a. arrow from $\vec{x}$ to $\vec{y}$ is the set of all points of $\mathbb{R}^n$ of the form $\vec{z}(t) = (1 - t)\vec{x} + t\vec{y}$ where $0 \leq t \leq 1$.

In this case the **tail** of the arrow will be at $\vec{x}$ and occur when $t = 0$ and $\vec{z}(0) = \vec{x}$ and the **head** or **tip** of the arrow will be at $\vec{y}$ and occur when $t = 1$ and $\vec{z}(1) = \vec{y}$.

NOTE: the function $\vec{z}(t)$ is our first example of a vector function of a scalar variable. It is also our first use of the parametric variable t. We shall be doing a lot of work with parametric representations of curves this semester!