Test 2: Multivariable Calculus

Math 224
Ron Buckmire

Friday April 16 2004
2:30pm-3:30pm

Name: ______________________

Directions: Read *all* problems first before answering any of them. Questions 1 and 2 are related. This is a one hour, open-notes, open book, test. **No calculators.** You must show all relevant work to support your answers. Use complete English sentences and CLEARLY indicate your final answers to be graded from your "scratch work."

No.	Score	Maximum
1		20
2		20
3		20
4		20
5		20
Extra Credit		10
Total		**100**

1. (*20 points.*) **Chain Rule, Implicit Function Theorem.**
Consider a surface implicitly-defined as $F(x, y, z) = 0$ which can be written as $z = f(x, y)$ so that $F(x, y, f(x, y)) = 0$.

a. (*10 points*) Use the Chain Rule to show that $\dfrac{\partial F}{\partial x} + \dfrac{\partial F}{\partial z}\dfrac{\partial z}{\partial x} = 0$ and $\dfrac{\partial F}{\partial y} + \dfrac{\partial F}{\partial z}\dfrac{\partial z}{\partial y} = 0$

b. (*10 points*) Use the implicit function theorem to obtain the equivalent result,

that is, $\dfrac{\partial z}{\partial x} = -\dfrac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial z}}$ and $\dfrac{\partial z}{\partial y} = -\dfrac{\dfrac{\partial F}{\partial y}}{\dfrac{\partial F}{\partial z}}$

2. (*20 points.*) **Partial Differentiation, Gradient Operator.**
Consider $\cos(x+y+z) = xyz$ as an example of $F(x,y,z) = 0$ and $z = f(x,y)$ from Question 1.
a. (*10 points*) Write down F and f, if you can. (If you can't, explain why the requested expression can not be obtained.) How are these two functions different?

b. (*10 points*) Write down $\vec{\nabla}F$ and $\vec{\nabla}f$, if you can. (If you can't, explain why the requested expression can not be obtained.) How are these two **vectors** different?

3. (*20 points.*) **Iterated Integration.**

a. (*10 points*) Evaluate $\displaystyle\int_{-3}^{0}\int_{0}^{2}\int_{-1}^{1} \cos(x+y+z) - xyz\ dx\ dz\ dy$

b. (*10 points*) Evaluate $\displaystyle\int_{1}^{2}\int_{0}^{\ln x} \frac{1}{x}\ dy\ dx$

4. (*20 points.*) **Multiple Integration.**

a. (*10 points*) Evaluate $\displaystyle\iint_R ye^x \, dA$ where R is the first quadrant of the circle of radius 4 centered at the origin. (Sketch the region R).

b. (*10 points*) Consider $\displaystyle\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} dz\, dy\, dx = \frac{1}{12}$. Re-compute this integral using a different triple integral which represents the same volume.

5. (*20 points.*) **Constrained Multivariable Optimization, Lagrange Multipliers**
The "geometric mean" of n numbers is defined as $f(x_1, x_2, \ldots, x_n) = \sqrt[n]{x_1 x_2 x_3 \ldots x_n}$. Suppose that $x_1, x_2, \ldots, x_n$ are positive numbers such that $\sum_{i=1}^{n} x_i = x_1 + x_2 + x_3 + \ldots x_n = c$, where c is a constant.

a. (*10 points*) Find the maximum value of the geometric mean of n positive numbers given the constraint that their sum must be equal to a constant. [HINT: Consider f^n instead of f!]

b. (*10 points*) You can deduce from part (a) that the geometric mean of n numbers is always less than or equal to the arithmetic mean, that is:

$$\sqrt[n]{x_1 x_2 x_3 \ldots x_n} \leq \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}$$

Under what conditions will the geometric mean be exactly equal to the arithmetic mean of those same n numbers?

EXTRA CREDIT (*10 points.*) **Unconstrained Multivariable Optimization**
Consider $f(x, y) = x^4 + y^4 - 4xy + 1$.
a. (*5 points*) Find the three critical points of $f(x, y)$.

b. (*5 points*) Use the Second Derivative Test to classify each of the three critical points of $f(x, y)$.