Report on Test 1

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Point Distribution (N=20)

Range	100+	93+	90+	87+	83+	80+	77+	73+	70+	67+	63+	60+	60-
Grade	A+	A	A-	B+	B	B-	C+	C	C-	D+	D	D-	F
Frequency	2	4	0	1	1	2	0	2	2	2	2	0	2

Summary Overall class performance was surprising (to me). The mean *and* median score were both 80. The standard deviation was 15. This means that exactly half of the class scored above 80 and half scored below.

#1 Equation of Planes, Vector Operations. The idea behind this question was similar to **Bonus Quiz #2**. It steps you through the procedure for finding the equation of a plane, given three points A, B and C. It asks for interpretation of the meaning of your calculation at most steps. **(a)** The vector from point A to B is $\vec{v} = B - A$ and from A to C is $\vec{w} = C - A$. **(b)** The dot product $\vec{v} \cdot \vec{w}$ ends up being zero, which means that $\vec{v}$ and $\vec{w}$ are orthogonal to one each other, or BAC is a right-angled triangle. **(c)** $\vec{v} \times \vec{w}$ ends up being $(-2, -5, 1)$ which is orthogonal to both $\vec{v}$ and $\vec{w}$. The magnitude of this vector is exactly twice the area of triangle BAC since it represents the area of the parallelogram between the two vectors. **(d)** The equation of the plane will be $-2(x - x_0) - 5(y - y_0) + 1(z - z_0) = 0$ where (x_0, y_0, z_0) are the coordinates of A, B or C. You could also say that the equation of the plane is $\vec{x} = s\vec{v} + t\vec{w} + \vec{A}$ where $\vec{A}$ is the vector corresponding to the coordinates of the point A. It must be the point A because $\vec{v} = \vec{B} - \vec{A}$ and $\vec{w} = \vec{C} - \vec{A}$.

#2 Level Sets, Vertical Slices. This question is to test visual and verbal skills. **(a)** Clearly Figure 1 is different from Figure 2 and Figure 3 so it is likely to be the level sets of $f(x, y) = x^3 - y^3$. When you notice that if $0 = k = f(x, y) = x^3 - y^3$ it will contain the line $y = x$ it confirms that Figure 1 represents the $z = k$ level sets. Figure 2 just looks like a cubic function shifted up and down, which would correspond to $z = f(x, y = k) = x^3 - k^3$, i.e. y is held constant. That leaves Figure 3 to be vertical slices with x held constant. Another way to confirm your ideas is to notice that as the horizontal input increases in value, in Figure 2 the vertical output increases also, supporting a standard cubic curve, which in Figure 3 it behave like a negative cubic curve. There was 6 points for making the correct identifications, 4 points for your explanations. **(b)** There's 5 points for just showing that you can take the partial derivatives of $f(x, y)$ to find f_x, f_y, f_{xy} and f_{yx}. The other 5 points are for noticing that near $(0, 0)$ in both Figure 2 and Figure 3 the slope of the curves is zero since the curves are parallel to the horizontal axis. This means that f_y and f_x are both zero at $(0, 0)$. However, this fact does not guarantee that f_{xy} and f_{yx} are zero! (Think about the function $f(x, y) = xy$ for example.) Near the origin the curves in Figure 2 and Figure 3 look identical if they are shifted vertically. This information tells you that both the rate of change of f_x with respect to y, i.e. f_{xy}, and the rate of change of f_y with respect to x, i.e. f_{yx} can be estimated to be zero near the origin. Actually it's true everywhere, but that is hard to tell from the graphs themselves at other points –but it is immediately apparent if you compute the "mixed partial derivatives" of $f(x, y)$.

#3 Tangent Plane Approximation. This is like BONUS Quiz #4. It is a very computation-oriented problem. **(a)**. Using the formula for a tangent plane one obtains the result that $T(x, y) = f(1, 2) + f_x(1, 2)(x-1) + f_y(1, 2)(y-2)$ which is equal to the tangent plane $z = 3x - 12y + 14$ **(b)** To estimate $0.9^3 - 1.99^3$ one is asking for an estimate of $f(0.9, 1.99)$ so one could just estimate by evaluating the tangent plane at this value. $f(.9, 1.99) \approx T(0.9, 1.99) = -7 + 3 \cdot (-0.1) - 12 \cdot (-0.01) = -7 - 0.3 + 0.12 = -7.18$ which is pretty close to the given exact value.

#4 Gradient, Directional Derivative. Notice the little headings at the top of each question (and in this Exam Report)? That's a hint that that topic will appear somewhere in that question! This problem is about using previously known concepts and applying them to an unfamiliar context. **(a)** This is just asking for the value of $\vec{\nabla} f$ evaluated at $(1, 2)$. That is, a vector representing the maximum rate of change of $f(x, y)$. Since $f(x, y) = x^3 - y^3$, $\vec{\nabla} f = (3x^2, -3y^2)$ which evaluated at the point (1,2) turns out to be the vector $(3, -12)$. (An aside): Please note that we are using the notation (a, b) to both mean the coordinates of a point in the xy-plane **and** the position vector $a\vec{i} + b\vec{j}$, depending on the context. This is a "feature" of the textbook, but not an uncommon one among mathematicians to choose a notation which must be interpreted based on the context in which it used. (End of aside) The magnitude of this vector $(3, -12)$, or $3\vec{i} - 12\vec{j}$, is the magnitude of how fast $f(x, y)$ is changing with respect to both its variables x and y at the specific input value $(1, 2)$. This is what the gradient tells you. That magnitude ends up being $\sqrt{3^2 + (-12)^2} = \sqrt{9 + 144} = \sqrt{153}$. I don't think it's unreasonable to ask you to do this kind of arithmetic without a calculator. Unreasonable would be asking for the value of $\sqrt{153}$ to two decimal places! (But you should think about how you would estimate that number given your knowledge of the square root function.) **(b)** This question asks you to tell them the rate of change of the function $f(x, y)$ in a particular direction $\vec{w}$ at the point $(1, 2)$ instead of the maximal direction computed in part (a). This is known as the directional derivative and denoted by $\dfrac{\partial f}{\partial \vec{w}}$ and computed by $\vec{\nabla} f(1, 2) \cdot \vec{w} = (3, -12) \cdot (4/5, -3/5) = 48/5$. All dot products are scalars. So, this is saying the rate of change of f in a particular vector direction is $48/5$. **(c)** This question asks you to choose a direction in which the rate of change would be zero. Lots of people realized that this must be a vector $\vec{w}$ which is orthogonal to $(3, -12)$. This is good. Unfortunately, some people chose the vector $(0, 0)$. While it is true this vector is orthogonal to $(3, -12)$ it is not a direction!! The direction $(4, 1)$, or equivalently $\dfrac{4}{\sqrt{17}}\vec{i} + \dfrac{1}{\sqrt{17}}\vec{j}$ would work.

#BONUS Continuity, Set Theory. To get bonus points you have to be even more precise than you should normally be about your answers to insure that you are communicating your reasoning and understanding clearly. The domain of the function $g(x, y)$ is the set of all possible input values to $g(x, y) = \dfrac{x^3 - y^3}{x - y}$. This is the set of all coordinate pairs (x, y) in "2-space" or $\mathbb{R}^2$ such that $x - y \neq 0$. It is absolutely not sufficient to say $x \neq y$ without stating what space the domain is a subset of. English sentences are useful here, or precise mathematical symbols. $D = \{(x, y) \in \mathbb{R}^2 | x \neq y\}$. A sketch would be helpful. The domain D is **open** since every point in it is an interior point, i.e. each point has a neighborhood of points around it which only contains points of the set. The points on $y = x$ are limit points of D but since they are specifically excluded from D, then D is **not closed** since it does not contain all its limit points. The function $g(x, y)$ is continuous on D because since $x \neq y$ and recalling the difference $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$ one can simplify $g(x, y)$ to be $g(x, y) = x^2 + xy + y^2$. Clearly a polynomial function in x and y must be continuous everywhere. It's not necessary to take any limits, and particularly not at the origin, since the origin is not in the domain D in the first place! Another way to know $g(x, y)$ is continuous is to show that its partial derivatives g_x and g_y exist and are continuous. FYI, differentiability is a STRONGER property than continuity, so if $g(x, y)$ is differentiable (i.e. $\vec{\nabla} g$ exists) everywhere then g is continuous everywhere.