Final Exam

Math 212 Fall 2005 **Multivariable Calculus**

Friday, December 9, 2005: 8:30–11:30am Prof. R. Buckmire

Name: ____________________

Directions: Read *all* problems first before answering any of them. There are TEN (10) problems on ELEVEN (11) pages.

This exam is an open-notes, open-book, test. You may **NOT** use a calculator. You must include ALL relevant work to support your answers. Use complete English sentences and CLEARLY indicate your final answer from your "scratch work."

No.	Score	Maximum
1.		**20**
2.		**20**
3.		**20**
4.		**20**
5.		**20**
6		**20**
7.		**20**
8.		**20**
9.		**20**
10.		**20**
TOTAL		**200**

1. *[20 points total.]* **Vector Operations, Equations of Lines.**
Consider the two vectors $\vec{\mathbf{a}} = (1, -2, 1)$ and $\vec{\mathbf{b}} = (-2, 1, 1)$.
(a) (*4 points.*) Compute $\vec{a} \cdot \vec{b}$.

(b) (*4 points.*) Compute $\vec{a} \times \vec{b}$.

(c) (*4 points.*) Find the coordinates of the midpoint between **a** and **b**.

(d) (*4 points.*) Write down the vector equation of the line passing through **a** and **b**.

(e) (*4 points.*) Is the point $(-5, 4, 1)$ on the line passing through **a** and **b**? How do you know? **Explain your answer!**

2. *[20 points total.]* **Equations of Planes, Distance**.

(a) (*5 points.*) Find the (shortest) distance between the planes $2x - y + 3z = 4$ and $2x - y + 3z = 6$. **Explain your answer!**

(b) (*5 points.*) Find the (shortest) distance between the plane $2x - y + 3z = 4$ and the point $(2, 3, 1)$. **Explain your answer!**

(c) (*10 points.*) Find the (shortest) distance between the planes $2x - y + 3z = 4$ and $2x + y - z = 4$. **Explain your answer!**

3. *[20 points total.]* **Multivariable Limits and Partial Derivatives.**
Evaluate the following limits.

(b) (*5 points.*) $\lim_{(x,y)\to(0,0)} \frac{x^3 - y^3}{x^3 + y^3}$. **Explain your answer!**

(b) (*5 points.*) $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2}$ **Explain your answer!**

(c) (*10 points.*) Find all the first partial derivatives of $f(x, y, z, t) = x^{y/z}$.

4. *[20 points total.]* **Gradient, Directional Derivatives and Tangent Planes.**
Consider the function $f(x,y) = \ln\sqrt{x^2+y^2}$.
(a) (*5 points.*) Show that $f_{xy} = f_{yx}$ for this function.

(b) (*5 points.*) Compute $\vec{\nabla} f$ at $(3,4)$.

(c) (*5 points.*) Compute the directional derivative of $f(x,y)$ in the direction $\vec{v} = -1\hat{\imath} - 2\hat{\jmath}$ at the point $(3,4)$.

(d) (*5 points.*) Use your answers above to approximate the value of $f(3.01, 3.97)$. (**Do not attempt to evaluate any logarithms**).

5. *[20 points total.]* **Multivariable Chain Rule.**
Consider two differentiable functions g and f where $g(s,t) = f(x,y)$ and $x = s^2 - t^2$ and $y = t^2 - s^2$. We want to use the Chain Rule to show that

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0.$$

(a) (*5 points.*) Draw the tree diagram for the function g.

(b) (*10 points.*) Write down expressions for $\frac{\partial g}{\partial s}$ and $\frac{\partial g}{\partial t}$.

(c) (*5 points.*) Show that $t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0$.

6. *[20 points total.]* **Unconstrained Multivariable Optimization.**
Find the critical points of the $f(x, y) = x^3 + y^3 - 6xy + 1$, and use the second derivative test to classify any local extrem values. What are the *global* extreme values of this function?

7. *[20 points total.]* **Constrained Multivariable Optimization, Lagrange Multipliers** Find the points on the ellipse $5x^2 - 6xy + 5y^2 = 4$ which are closest to and furthest from the origin. (**HINT: Optimize the square of the distance between points on the curve and the origin!**)

8. *[20 points total.]* **Iterated Integration.**

(a) (*6 points.*) $\displaystyle\int_0^{\ln 2}\int_0^{\ln 5} e^{2x-y}\,dx\,dy$

(b) (*6 points.*) $\displaystyle\int_0^{1}\int_{x^2}^{1} x^3\sin(y^3)\,dy\,dx$

(c) (*8 points.*) $\displaystyle\int_0^{1}\int_{x}^{2x}\int_0^{y} xyz\,dz\,dy\,dx$

9. *[20 points total.]* **Green's Theorem.**

By evaluating the line integral $\frac{1}{2}\oint_{\Gamma} x\,dy - y\,dx$ and applying Green's Theorem, we want to show that the area of an ellipse is πab where $2a$ and $2b$ are the lengths of the minor and major axes of the ellipse and Γ is the closed path in $\mathbb{R}^2$ traced out by the ellipse, which is centered about the origin.

(a) (*6 points.*) Write down a multiple integral which represents the area of the ellipse. (Draw a picture!) **DO NOT EVALUATE THE INTEGRAL!**

(b) (*10 points.*) Evaluate the line integral $\frac{1}{2}\oint_{\Gamma} x\,dy - y\,dx$ where Γ is the closed path traced out by the ellipse in the counter clockwise direction.

(c) (*4 points.*) Explain how your above work allows you to find the area of the ellipse to be πab.

10. *[20 points total.]* **Gradient Fields, Div, Grad and Curl.**
Consider the vector field $\vec{F}(\vec{x}) = F_1(x,y,z)\hat{\imath} + F_2(x,y,z)\hat{\jmath} + F_3(x,y,z)\hat{k}$ where $\vec{x}$ is in $\mathbb{R}^3$. Recall, all gradient fields have zero curl. Recall also, a symmetric matrix is one in which $A_{ij} = A_{ji}$, in other words the transpose of matrix A equals matrix A. For example, $\begin{bmatrix} 1 & 0 & -7 \\ 0 & 2 & 4 \\ -7 & 4 & 3 \end{bmatrix}$ is symmetric.

(a) (*10 points.*) Show that if $\vec{F}$ is a **gradient field**, then the Jacobian of $\vec{F}$ is a **symmetric matrix**.

(b) (*10 points.*) Show that if the Jacobian of $\vec{F}$ is a **symmetric matrix**, then $\vec{\nabla} \times \vec{F} = \vec{0}$.