Convergence Tests for Infinite Series

I. n-th Term Test for Divergence

This test comes about from the definition of what convergence of a series means.

> If $\lim_{n\to\infty} a_n \neq 0$ (or does not exist) then the series $\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + a_4 + \cdots$ **diverges**.

Associated with the above fact is the idea that **IT IS TRUE** that if $\sum_{k=1}^{\infty} a_k$ CONVERGES, then $\lim_{n\to\infty} a_n = 0$.

We know from class is that **IT IS NOT TRUE** that if $\lim_{n\to\infty} a_n = 0$ then $\sum_{k=1}^{\infty} a_k$ CONVERGES.

II. Integral Test for Convergence and Divergence

This test relates facts about improper integrals to facts about infinite series.

> Suppose $f(x)$ is a continuous and decreasing function and $f(x) > 0$ for all $x \geq 1$. Let $a(k) = f(k)$. THEN
>
> (a) If the $\int_1^{\infty} f(x)dx$ CONVERGES, then the infinite series $\sum_{k=1}^{\infty} a_k$ CONVERGES.
>
> (b) If the $\int_1^{\infty} f(x)dx$ DIVERGES, then the infinite series $\sum_{k=1}^{\infty} a_k$ DIVERGES.

Definition For $p > 0$ the infinite series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ is called a p-series.

> If one applies the integral test to the p-series then
>
> if $p \leq 1$, then p-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ DIVERGES.
>
> If $p > 1$, then the p-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ CONVERGES.

III. Comparison Test for Convergence and Divergence

(a) If $0 \le b_k \le a_k$ for each k and $\sum_{k=1}^{\infty} a_k$ converges, then $\sum_{k=1}^{\infty} b_k$ also CONVERGES.

(b) If $0 \le a_k \le c_k$ for each k and $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} c_k$ also DIVERGES.

IV. Alternating Series Test

Definition An infinite series is said to be an **alternating series** if it can be written in the form $\sum_{k=1}^{\infty} (-1)^k a_k$ or $\sum_{k=1}^{\infty} (-1)^k a_k$ where $a_1; a_2; a_3; \ldots$ are all positive numbers.

If the terms of the series are (i) decreasing in magnitude and (ii) $\lim_{n \to \infty} a_n = 0$ then the alternating series $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ CONVERGES.

V. Absolute Ratio Test

For any infinite series $\sum_{k=1}^{\infty} a_k$, if

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$$

then $\sum_{k=1}^{\infty} a_k$ CONVERGES.

If $L > 1$ or if $|a_{k+1}/a_k|$ does not exist, then $\sum_{k=1}^{\infty} a_k$ DIVERGES

If $L = 1$ the test is INCONCLUSIVE.

VI. Limit Comparison Test

Let $\sum_{k=1}^{\infty} a_k$, be an infinite series of **positive** terms.

(a) If $\sum_{k=1}^{\infty} c_k$ is a **convergent** series of positive terms, and $\lim_{n \to \infty} \frac{a_n}{c_n}$ EXISTS and is NOT INFINITE then $\sum_{k=1}^{\infty} a_k$ also CONVERGES.

(b) If $\sum_{k=1}^{\infty} c_k$ is a **divergent** series of positive terms, and $\lim_{n \to \infty} \frac{a_n}{d_n}$ EXISTS and is NOT ZERO or IS INFINITE then $\sum_{k=1}^{\infty} a_k$ also DIVERGES.