Math 120 Spring 1999 Final Review Assignment and Topics Summary

Ron Buckmire
Ramin Naimi

1. **The ideas are the most important thing!**

2. **Practice some techniques.** Important techniques include determining the number of subdivisions needed to obtain a Riemann sum approximation of the definite integral of a monotone function to a given degree of accuracy; finding the derivative of an accumulation function; relating the graphs of a function, its derivative and its family of antiderivatives; writing the solution of an initial value problem as an accumulation function; using basic properties of integrals and antiderivatives; and using the Fundamental Theorem of Calculus to evaluate definite integrals. In particular, you should know the table of antiderivatives below.

a.

$f(x) = F'(x)$	$F(x) = \int f(x)dx$
1	$x + C$
x^n $(n \neq -1)$	$\frac{1}{n+1}x^{n+1} + C$
$\frac{1}{x}$	$\ln\lvert x\rvert + C$
$\sin(x)$	$-\cos(x) + C$
$\cos(x)$	$\sin(x) + C$
$\sec^2(x)$	$\tan(x) + C$
$\frac{1}{1+x^2}$	$\arctan(x) + C$
e^x	$e^x + C$
$a^x (a > 0)$	$\frac{1}{\ln(a)}a^x + C$
$\tan(x)$	$-\ln(\lvert\cos(x)\rvert) + C$
$\ln(x)$	$x\ln(x) - x + C$

b. finding antiderivatives and evaluating definite integrals by using u-substitution ($\int f'[g(x)]g'(x)\,dx = \int f'[u]\,du = f[u] + C = f[g(x)] + C$ pick $u = g(x)$ and convert integral to u variables. You can either return to x-space or stay lost in u-space when evaluating definite integrals); However You should be able to convert one definite integral in x-space to another integral in u-space given the particular u-substitution

c. finding antiderivatives and evaluating definite integrals by using integration by parts ($\int f \cdot g'\,dx = f \cdot g - \int f' \cdot g\,dx$ or $\int u\,dv = uv - \int v\,du$). Don't forget

(i) Repeated integration by parts e.g. $\int x^4 e^{-x}dx$

d. finding the area between two curves by setting up the proper integral over the proper interval and evaluating it using the fundamental theorem of calculus or approximating it using numerical methods;

e. finding the average value of a function on a given interval by setting up the proper integral over the proper interval, i.e. $\frac{1}{b-a}\int_a^b f(x)dx$

f. using basic algebra skills and trigonometric identities to help simplify integrands, so that one may find antiderivatives and evaluate definite integrals;

g. using numerical methods of integration:

- Riemann Sums $\sum_{k=1}^{N} f(x_k)\Delta x$
- Left Hand Sums $x_k = a + (k - 1) \cdot \Delta x$, Right Hand Sums $x_k = a + k \cdot \Delta x$
- Midpoint method $x_k = a + (k - .5) \cdot \Delta x$
- Trapezoidal Rule $T = \frac{L+R}{2}$
- Simpson's Rule $S = \frac{2}{3}M + \frac{1}{3}T$
- Midpoint Error versus Trapezoid Error (depends on concavity f'' and N^{-2})
- Left Riemann Error versus Right Riemann Error (depends on slope f' and N^{-1})
- Simpson Error (depends on $f^{(4)}$ and N^{-4})
- Error Control e.g. Riemann error $= |f(b) - f(a)|\frac{b-a}{N} \leq .001 \Rightarrow$ solve for N

h. using the fundamental theorem of calculus to compute an arclength of a function $f(x)$ on an interval $[a, b]$ by setting up the function $g(x) = \sqrt{1 + [f'(x)]^2}$ and integrating over the given interval.

3. Other topics include:

a. improper integrals of the first kind and of the second kind

b. determining convergence of improper integrals using the Comparison Rule

c. polynomial approximations of a function near a point (Taylor polynomials), applications of Taylor polynomial approximations to derivatives and anti-derivatives, using calculus and algebra to find new Taylor Series from familiar ones

d. trigonometric approximations to periodic functions (Fourier Polynomials), computing Fourier Series coefficients using integral formulas, remembering properties of odd/even functions

e. tests for convergence of infinite series **n-th term (zero-limit), alternating series, integral, basic comparison, absolute ratio and limit comparison**

f. useful series to remember are p-series, geometric series, harmonic series, alternating harmonic series

g. Remember the differences and connections between improper integrals and infinite series and be able to articulate these in written form

4. **Practice using tests for convergence.** Especially important are the Absolute Ratio Test and the n-th Term Test. Don't come into the exam without being able to take the limit as $k \to \infty$ of some expression involving k. You should be able to apply L'Hopital's rule on those indeterminate limits. Don't forget the other tests we have covered (the Integral Test, the Basic Comparison Test, Limit Comparison, Absolute Comparison and Alternating Series Test).

5. Remember the basic idea of doing comparisons:
   If you want to show that something CONVERGES, you have to compare it to something which is **LESS THAN OR EQUAL TO** something you already know CONVERGES.
   If you want to show that something DIVERGES, you have to compare it to something which is **GREATER THAN OR EQUAL TO** something you already know DIVERGES.
   The "something" can either be an improper integral or an infinite series, but in either case the integrand or terms must all be POSITIVE. FUNCTIONS DO NOT converge or diverge, improper integrals or infinite series do.

6. Taylor Series To Remember...

$$\begin{aligned}
\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots &&= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \\
\cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots &&= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \\
\tfrac{1}{1-x} &= 1 + x + x^2 + x^3 + \cdots &&= \sum_{k=0}^{\infty} x^k \\
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots &&= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} \\
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots &&= \sum_{k=0}^{\infty} \frac{x^k}{k!}
\end{aligned}$$

7. **Evaluating Limits**
   L'Hôpital's Rule
   If $\lim_{x \to \infty} \frac{f(x)}{g(x)}$ is of the form $\frac{\infty}{\infty}$ or $\frac{0}{0}$ or $0 \cdot \infty$ then
   if if the limit $\lim_{x \to \infty} \frac{f'(x)}{g'(x)} = L$ exists, then $\lim_{x \to \infty} \frac{f(x)}{g(x)} = L$ In other words, if you have an indeterminate limit, just differentiate the numerator and denominator and take the limit again until you get a determinate answer. That answer will be the value of the limit.
   You should be comfortable with discounting or ignoring parts of an expression when these parts are getting very small compared to the rest of the expression. Remember, sin(x) and cos(x) only return values between ± 1.
   Recall the various rules involving limits, such as $\lim_{b \to \infty} b^p$, $\lim_{x \to \infty} e^{kx}$
   Remember that when there's a race between e^x and any polynomial function x^p as $x \to \infty$, e^x will always win. Conversely, ln(x) will lose any race with x^p as $x \to \infty$

8. **Interval Of Convergence and Radius Of Convergence**

Consider $\sum_{k=0}^{\infty} b_k(x-a)^k$. This Power Series may not converge for all x-values. The set of x-values for which the series converges is called the interval of convergence. The interval of convergence is always centered on the point a.

The interval of convergence can be infinite, i.e. $(-\infty, \infty)$ a.k.a. "all Real Numbers". Or it can be a finite interval of the form $(a-R, a+R), [a-R, a+R], (a-R, a+R]$ or $[a-R, a+R)$. R is called the radius of convergence. The value $R = 1/L$ and is computed using the absolute ratio test.

$$\frac{1}{R} = L = \lim_{k \to \infty} \left| \frac{b_{k+1}}{b_k} \right|$$

BLUE NOTES

Name: ______________________________

SAMPLE PROBLEMS FOR FINAL EXAM

1. Draw the area represented by the following integral

$$\int_0^4 \frac{1}{1+x^2}\,dx.$$

Using n = 4 subintervals, estimate the definite integral using the following:

a. left endpoint Riemann sum
b. right endpoint Riemann sum
c. midpoint Riemann sum
d. trapezoid rule and
e. Simpson's rule.
f. Compute the exact integral (use the FTC).

Then compare your estimates. Know which ones are most accurate and why.

2. a. Using integration by substitution, find $\int x\sqrt{1+3x}\,dx$

b. Using integration by parts, find $\int \frac{\ln(x)}{x^2}\,dx$

c. Using integration by parts or integration by substitution, find $\int \sin^2(x)\cos x\,dx$

d. By evaluating a definite integral, find the area under the x-axis but above the curve $y = x^2 - 3x$. Draw a sketch of the curve and indicate the requested area on your sketch.

e. Below is a list of indefinite integrals. Find an antiderivative for each.

$$\int \sin(2x)dx; \quad \int 3\cos^2(4x)\sin(4x)\,dx; \quad \int 7^x\,dx; \quad \int x^2(x^3-6)^{20}\,dx; \quad \int \frac{3}{4x-2}\,dx.$$

f. Find the average value of $f(x) = \sin^2(3x)\cos(3x)$ on $[0, \frac{\pi}{6}]$.

g. Evaluate the following: $$\int_0^1 \frac{x}{\sqrt{1+x}}\,dx; \qquad \int \frac{x}{1+\sqrt{x}}\,dx.$$

3. Explain why using Simpson's method to evaluate the definite integral in part (d) above will compute the answer **exactly**, but if you were to use the Midpoint or Trapezoid Method the answer would only be approximate. (You can test this for yourself by trying to evaluate the definite integral using Midpoint, Trapezoid and Simpson's Method and seeing that Simpson's is exact.)

4. Calculate the following (improper) integrals.

   a. $\displaystyle\int_0^1 \ln(2x)\,dx$

   b. $\displaystyle\int_0^6 \frac{1}{\sqrt{6-x}}\,dx$

   c. $\displaystyle\int_3^8 \frac{1}{2-x}\,dx$

   d. $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}+2}\,dx$

   e. $\displaystyle\int_1^\infty \frac{x^2}{1+\sin^2(x)}\,dx$

   f. $\displaystyle\int_1^\infty \frac{\sin^2(x)}{1+x^2}\,dx$

5. Calculate the area of the region under the curve $y = \sqrt{x} + 1$, above the x-axis and between $x = 0$ and $x = 4$.

6. Find the unique solution to the IVP

$$y' + y(1 + x) = 0; \qquad y(-2) = 1$$

7. Given that $A(X) = \int_1^{x} \sqrt{1 + e^{2x}}\,dx$

   a. Evaluate $A(1)$
   b. Evaluate $A'(1)$
   c. Evaluate $A''(1)$
   d. Show that $A(b)$ represents the length of the curve $y = e^x$ from the coordinate $(1, e)$ to (b, e^b)
   e. If $B(X) = \int_1^{\sin(X)} \sqrt{1 + e^{2x}}\,dx$, Find $B'(X)$.

8. Compute the following:

   $A = \int_0^2 x^2 t\,dx$

   $B = \int_0^2 x^2 t\,dt$

   $C = \int_0^2 x^2 t\,dk$

   $D = \int_0^x k^2 t\,dt$

   $E = \int_0^k x^2 t\,dx$

   What is $\frac{dB}{dx}$ equal to? What about $\frac{dD}{dx}$? $\frac{dE}{dx}$? $\frac{dA}{dt}$?

9. Solve the following initial value problem

$$f'(x) = \frac{3}{x} + e^x + x^3; \qquad f(2) = 0.$$

10. a. Find the second degree Taylor polynomial based at $a = 2$ for the function $g(x) = e^{2-x}$.

    b. Use your answer in part (a) to estimate

$$\int_{1.9}^{2.1} e^{2-x}\,dx.$$

11. Determine whether the following series converge or diverge.

a. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3+4^n}$

b. $\displaystyle\sum_{n=0}^{\infty} \frac{6^n}{n!}$

c. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{5}{\sqrt{n^3}}$

d. $\displaystyle\sum_{n=1}^{\infty} (-1) \frac{2}{\sqrt{n}}$

12. Evaluate the following sums exactly:

$$1 + \frac{1}{e} + \frac{1}{e^2} + \frac{1}{e^3} + \cdots.$$

$$1 - \frac{1}{e} + \frac{1}{e^2} - \frac{1}{e^3} + - \cdots.$$

$$1 + 2\frac{1}{e} + 3\frac{1}{e^2} + 4\frac{1}{e^3} + \cdots.$$

$$1 + \frac{1}{e^2} + \frac{1}{e^4} + \frac{1}{e^6} + \cdots.$$

14. Write down the Taylor Series for $f(x) = e^{-x^3}$ about the point $x = -1$. (HINT: You probably do **not** want to do this by taking any derivatives of $f(x)$.)

15. Find the radius and interval of convergence of the infinite series $\displaystyle\sum_{k=0}^{\infty} (-1)^k x^{2k}$

16. Consider the function $f(x)$ is periodic with period pi.

$$f(x) = \begin{cases} x & -\pi \le x < 0 \\ -x & 0 \le x < \pi \end{cases}$$

(a) Sketch a graph of $f(x)$ on the interval $[-10, 10]$. Is $f(x)$ an even or odd function?
(b) Find all the coefficients of the Fourier Series for $f(x)$, a_k and b_k