Write down an example of a periodic function that you know and sketch it below:

\[ f(t) = \sin(t) \]

**Group Work**

Label each of the following as periodic or not periodic. If the function is periodic, find its period.

(a) \( f(x) = \sin(x) \)  
(b) \( g(t) = t^2 \)  
(c) \( f(x) = x^2 \sin(x) \)  
(d) \( f(t) = t \)  
(e) \( f(t) = 4 \)  
(f) \( h(x) = \cos(2x) \)  
(g) \( f(x) = \begin{cases} 2 & \text{if } 2n \leq x \leq 2n + 1 \\ 1 & \text{if } 2n + 1 < x < 2n + 2 \end{cases} \)  

where \( n \) is an integer

Pick which functions you think are periodic, sketch them below, and indicate their period.
Fourier Series

In general, a Fourier Series is used to approximate a function $f(t)$ with period $2\pi$

$$f(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(kt) + \sum_{k=1}^{\infty} b_k \sin(kt)$$

where

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \, dt$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(kt) \, dt$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(kt) \, dt$$

This usually involves a fair amount of integration to find explicit forms of the coefficients $a_k$ and $b_k$. **NOTE:** $a_0$ is the average value of $f(x)$ on the interval $[-\pi, \pi]$.

**EXAMPLE**

Consider the following function, which is a famous signal called a square wave.

$$f(x) = \begin{cases} 
-1 & \text{if } -\pi \leq x \leq 0 \\
1 & \text{if } 0 < x < \pi 
\end{cases}$$

1. Sketch the graph of $f(x)$ below between $-2\pi \leq x \leq 3\pi$.

2. Find the zeroth degree Fourier polynomial for $f(x)$.

2. Find the first degree Fourier polynomial for $f(x)$.  

Exercise

3. Show the general form of the Fourier polynomial for this $f(x)$ is $+ \sum_{k=1}^{\infty} \frac{2}{\pi} [1 - (-1)^k] \sin(kx)$.

4. Write down the 7th degree Fourier polynomial approximation to the square wave.

5. For what values of $x$ will the infinite series converge? What happens when you try absolute ratio test?
The Fourier polynomials $F_N(x) = \sum_{k=1}^{N} \frac{2}{N} [1 - (-1)^k] \sin(kx)]$ is graphed below. The figures show $F_1(x), F_7(x)$ and $F_{15}(x)$.

What do you think the graph of $F_{\infty}(x)$ looks like?