Introduction to Taylor Series and Maclaurin Series

Warm-Up

(a) What’s the equation of a tangent line to the function \( f(x) = e^x \) at \( x = 0 \)?

We can Represent ANY Function By A Power Series!

Let’s suppose we can represent the function \( f(x) \) by a power series centered at \( a \) (also known as the power series about \( a \))

\[
f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots
\]

Let’s take the first three derivatives of this function

\[
f'(x) = 0 \cdot c_0 + 1 \cdot c_1 + 2 \cdot c_2(x-a) + 3 \cdot c_3(x-a)^2 + 4 \cdot c_4(x-a)^3 + \ldots
\]

\[
f''(x) = 0 \cdot c_0 + 0 \cdot c_1 + 2 \cdot c_2 + 3 \cdot 2 \cdot c_3(x-a) + 4 \cdot 3 \cdot c_4(x-a)^2 + \ldots
\]

\[
f^{(3)}(x) = 0 \cdot c_0 + 0 \cdot c_1 + 0 \cdot c_2 + 3 \cdot 2 \cdot c_3 + 4 \cdot 3 \cdot 2 \cdot c_4(x-a) + \ldots
\]

Look at what happens when we evaluate these derivatives at the value \( x = a \),

\[
f'(a) = 1 \cdot c_1
\]

\[
f''(a) = 2 \cdot 1 \cdot c_2
\]

\[
f^{(3)}(a) = 3 \cdot 2 \cdot 1 \cdot c_3
\]

By remembering that \( f(a) = c_0 \) we can get an expression for the first four terms of the power series for \( f(x) \) centered about the point \( x = a \)

\[
c_0 = f(a)
\]

\[
c_1 = f'(a)
\]

\[
c_2 = \frac{f''(a)}{2}
\]

\[
c_3 = \frac{f^{(3)}(a)}{3 \cdot 2}
\]

\[
c_4 = \frac{f^{(4)}(a)}{4 \cdot 3 \cdot 2}
\]

\[
\vdots
\]

\[
c_n = \frac{f^{(n)}(a)}{n!}
\]

In other words, now that we have an expression for the \( n^{th} \) coefficient, we can represent the function \( f(x) \) by the following power series:

\[
f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \ldots = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k
\]

This expression is known as the Taylor Series (also known as the Taylor Series expansion) for the function \( f(x) \) about the point \( x = a \). It allows us to find a power series associated with any given function.
DEFINITION: MacLaurin Series
The Taylor Series expansion for a given function about the point \( a = 0 \) is called the MacLaurin Series for the function \( f(x) \).

\[
f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots = \sum_{k=0}^{\infty} \frac{f(k)(0)}{k!}x^k
\]

EXAMPLE
Let’s show that the Taylor Series expansion for \( f(x) = \sin(x) \) about the point \( a = 0 \) is

\[
\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}
\]

Let’s find the radius of convergence of the Maclaurin Series for \( \sin(x) \).

Exercise
Find the MacLaurin Series for \( f(x) = e^x \) and show that it converges to \( e^x \) for every \( x \)-value.
MacLaurin Series That We Should All Know

\[
\sin(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots
\]

\[
\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots
\]

\[
e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots
\]

\[
\arctan(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots
\]

\[
\ln(1 + x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots
\]

\[
\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \ldots
\]

\[
(a + x)^n = \sum_{k=0}^{\infty} \frac{a^{n-k}x^k}{k!(n-k)!} = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + \ldots
\]

**NOTE:** The first three of these have infinite radius of convergence, while the other have a radius of convergence of 1. (Their intervals of convergence may vary so you need to check the end points!)

**EXAMPLE**

What’s the Taylor Series Expansion of \(\ln(1 - x^2)\) and for what values of \(x\) is it valid?
**DEFINITION: Taylor Polynomial**

The $n^{th}$ degree Taylor Polynomial approximation for a given function $f(x)$ about the point $(a, f(a))$ is the partial sum of the $n + 1$ terms of the **Taylor Series** for the function $f(x)$ about the point $a$.

**EXAMPLE**

What’s the first order Taylor Polynomial approximation to $f(x) = e^x$ at $x = 0$? What’s the second-order Taylor Polynomial approximation?

The first order Taylor approximation of a function $f(x)$ at $x = a$ is equivalent to the tangent line approximation to $f(x)$ at $a$. 