Using Power Series To Represent Functions

RECALL

We showed last time that we could represent the function \( f(x) = \frac{1}{1-x} \) by the power series
\[
\sum_{n=0}^{\infty} x^n \quad \text{when} \quad -1 < x < 1.
\]
Can we do this for other functions? Sure!

Exercise

Let’s represent the function \( \frac{1}{1+x^2} \) by a power series. (Find the radius and interval of convergence of this power series.)

EXAMPLE

Remember \( \int \frac{1}{1+x^2} \, dx = \arctan(x) \) and \( \arctan(1) = \frac{\pi}{4} \).

We can use this information to show the amazing result

\[
\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \ldots = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \quad \text{(Leibniz \( \pi \) Formula)}
\]

So we have shown that \( \arctan(x) \) can be represented by a power series on the interval \(-1 \leq x \leq 1\).
**Theorem**

Given a power series \( \sum_{n=0}^{\infty} c_n(x-a)^n \) has radius of convergence \( R > 0 \), the function defined by \( f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \) possesses a derivative \( f'(x) \) and anti-derivative \( F(x) \) on the interval \((a - R, a + R)\) with

(i) \( f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \ldots = \sum_{n=0}^{\infty} c_n n(x-a)^{n-1} \)

(ii) \( F(x) = \int f(x) dx = C + c_0(x-a) + c_1 \frac{(x-a)^2}{2} + c_2 \frac{(x-a)^3}{3} + \ldots = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} \)

The radius of convergence of \( f'(x) \) and \( F(x) \) are both \( R \) (the same as the radius of convergence of \( f(x) \)). The intervals of convergence may differ however.

**Group Work**

Use this theorem to obtain a power series representation of \( \ln(1 + x) \). What are the interval of convergence and radius of convergence for the series.

**Exercise**

Stewart, page 475, #42. Find the sum of the series \( \sum_{n=1}^{\infty} \frac{4^n}{n5^n} \).