Fourier Series Approximations

§1 Fourier Series

Suppose that \( f(x) \) is a periodic function with period \( 2\pi \). If we want to approximate this function with a trigonometric polynomial of degree \( n \),

\[
F_n(x) = a_0 + a_1 \cos(x) + b_1 \sin(x) + a_2 \cos(2x) + b_2 \sin(2x) + \cdots + a_n \cos(nx) + b_n \sin(nx)
\]

then the “best” coefficients to use are the following Fourier coefficients:

\[
a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx
\]

\[
a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) \, dx
\]

\[
b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) \, dx
\]

where \( k \geq 1 \).

Whereas a Taylor Series attempts to approximate a function locally about the point where the expansion is taken, a Fourier series attempts to approximate a periodic function over its entire domain. That is, a Taylor series approximates a function pointwise and a Fourier series approximates a function globally.

Error in Using Fourier Series

The error in approximating a \( 2\pi \)-periodic function \( f(x) \) by a Fourier polynomial \( F_n(x) \) is given by the following integral:

\[
E_n = \int_{-\pi}^{\pi} |f(x) - F_n(x)|^2 \, dx.
\]

That is, \( F_n(x) \) is a “good” approximation of \( f(x) \) if \( E_n \) is small. Notice that \( E_n \) does not depend on \( x \). \( E_n \) gives a measure of how well \( F_n(x) \) approximates \( f(x) \) over all of \( [-\pi, \pi] \), not just at a single point as our error bound for Taylor polynomials did. The above Fourier coefficients are the “best” coefficients in the sense that those coefficients make \( E_n \) as small as possible.
Part 1: Computing Fourier series

Consider the sawtooth wave of period $2\pi$ given by $f(x) = x$ when $-\pi \leq x \leq \pi$. Give a sketch of the sawtooth wave from $-4\pi \leq x \leq 4\pi$.

Find the first three Fourier polynomials, $F_1(x), F_2(x),$ and $F_3(x)$.

(Hint: $a_0$ and the rest of the $a_k$'s are easy to determine with a bit of thought. Focus on the graph of the function and think about the area.)

Now write down a general formula for the Fourier coefficients $a_k$ and $b_k$. You may want to break the integral up into two parts to compute $b_k$. You can use Wolfram’s integration capabilities to check your answer.