2003 EXAM 2

4. *(20 points total.)* **Improper Integrals.** In each part of this question, you need to think up an example of a function whose improper integral has the required properties. In each case your function **MUST INCLUDE THE FUNCTION** $e^{\square}$ **SOMEWHERE(though it may also include other functions). You must also show or prove or explain WHY the improper integral you write down produces the correct, required answer.**

a. Write down an improper integral of the first kind which converges. **You must include the exponential function.**

$$\int_1^{\infty} e^{-x}\,dx \quad \text{converges}$$

$$\lim_{b\to\infty}\int_1^b e^{-x}\,dx = \lim_{b\to\infty} -e^{-x}\Big|_1^b$$

$$= \lim_{b\to\infty} -e^{-b} - (-e^{-1})$$

$$= \lim_{b\to\infty} -e^{-b} + \frac{1}{e} = \frac{1}{e} \checkmark$$

b. Write down an improper integral of the first kind which diverges. **You must include the exponential function.**

$$\int_1^{\infty} e^{x}\,dx \quad \text{DIVERGES}$$

$$\lim_{b\to\infty}\int_1^b e^{x}\,dx = \lim_{b\to\infty} e^{x}\Big|_1^b = \lim_{b\to\infty} e^{b} - e^{1}$$

$$= \infty$$

2003 EXAM 2

2. *(30 points total.)* **Separation of Variables.**

(a) *(20 points)* Find the exact solution to $\dfrac{dy}{dx} = y \cdot (\ln(x) + 1), \qquad y(1) = 1$

$$\frac{dy}{y} = \ln(x) + 1\, dx$$

$$\int \frac{dy}{y} = \int \ln x + 1\, dx$$

$$\ln y = x \ln x - x + x + C$$

$$\ln y = x \ln x + C$$

$x = 1, y = 1$

$$\ln 1 = 1 \cdot \ln 1 + C$$

$$0 = 0 + C$$

$$0 = C$$

$$\ln y = x \ln x = \ln(x^x) \Rightarrow y = x^x$$

(b) *(10 points)* Confirm that the answer to part (a) is $y(x) = x^x = e^{x \ln(x)}$

$x = 1, y = 1$

$1^1 = 1$ ✓

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{x \ln x}\right) = e^{x \ln x} \cdot (x \ln x)'$$

$$= x^x \left(x \cdot \frac{1}{x} + 1 \cdot \ln x\right)$$

$$= x^x (1 + \ln(x))$$

1999 EXAM 2

5. (20 points) Consider $f(x) = \arctan(x)$. We want to determine the function $F(x)$ which when differentiated, equals $\arctan(x)$. In other words, $F(x)$ is the **anti-derivative** of $\arctan(x)$.

a. Write down the **derivative** of $f(x) = \arctan(x)$, that is $f'(x)$ (You can purchase this answer from me for four points)

$$f'(x) = \frac{1}{1+x^2}$$

b. Using integration by parts (once!) on the function written as $1 \cdot \arctan(x)$ find $F(x) = \int 1 \cdot \arctan(x)\, dx$ (HINTS: be careful about your choice of du and v and you may have to use substitution AFTER you use parts)

$$I = \int 1 \cdot \arctan(x)\, dx = x \cdot \arctan(x) - \int x \cdot \frac{1}{1+x^2}\, dx$$

$u = \arctan(x) \qquad du = \frac{1}{1+x^2}\, dx$

$dv = 1 \cdot dx \qquad v = x$

$$I = x \cdot \arctan(x) - \frac{1}{2}\int 2x \cdot \frac{1}{1+x^2}\, dx$$

$$F(x) = x \cdot \arctan(x) - \frac{1}{2}\ln(1+x^2) + C$$

c. Verify you have indeed anti-differentiated correctly by computing $F'(x)$

$$F'(x) = 1 \cdot \arctan(x) + x \cdot \frac{1}{1+x^2} - \frac{1}{2} \cdot \frac{1}{1+x^2} \cdot 2x$$

$$= \arctan(x) + \frac{x}{1+x^2} - \frac{x}{1+x^2}$$

$$= \arctan(x)$$

1998 EXAM 2

4. (*20 points*) Find the average value, $\overline{g}$, of the function $g(r) = r\ln r - r$ on the interval $[1, e]$.

$$\overline{g} = \frac{1}{e-1}\int_1^e r\ln r - r\,dr$$

$$= \frac{1}{e-1}\int_1^e r\ln r\,dr - \frac{1}{e-1}\int_1^e r\,dr$$

$$= \frac{1}{e-1}\left(\frac{r^2}{2}\cdot\ln r\right)\Big|_1^e - \frac{1}{e-1}\int_1^e \frac{r^2}{2}\cdot\frac{1}{r}\,dr - \frac{1}{e-1}\,\frac{r^2}{2}\Big|_1^e$$

$u = \ln r$ $\quad du = \frac{1}{r}dr$

$dv = r$ $\quad v = \frac{r^2}{2}$

$$\overline{g} = \frac{1}{e-1}\left(\frac{e^2}{2}\ln e - \frac{1}{2}\ln 1\right) - \frac{1}{e-1}\int_1^e \frac{r}{2}\,dr - \frac{1}{e-1}\,\frac{e^2-1}{2}$$

$$= \frac{1}{e-1}\left(\frac{e^2}{2}\cdot\ln e - \frac{1^2}{2}\cdot\ln 1\right) - \cancel{\ldots} \frac{1}{e-1}\,\frac{2}{2}$$

$$= \frac{1}{e-1}\cdot\frac{e^2}{2} \cancel{+\frac{3(e-1)}{4}} - \frac{1}{e-1}\,\frac{(e^2-1)}{4} - \frac{1}{e-1}\,\frac{(e^2-1)}{2}$$

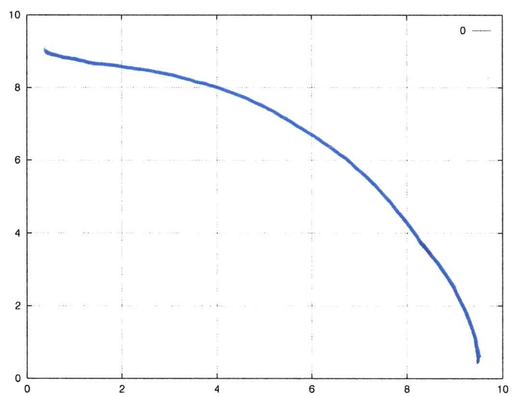
$$= \frac{1}{e-1}\left[\frac{e^2}{2} - \frac{e^2}{4} + \frac{1}{4} - \frac{e^2}{2} + \frac{1}{2}\right]$$

$$= \frac{1}{e-1}\left[\frac{3}{4} - \frac{e^2}{4}\right]$$

1998 EXAM 2

3. (*20 points*) Let **I** represent the exact value of $\int_0^{10} f(x)\,dx$ (where $f(x) \geq 0$ on $[0, 10]$) and **T**, **L**, **R** and **M** represent, respectively, the Trapezoid approximation, Left-Hand Riemann Sum, Right-hand Riemann sum and Midpoint Riemann Sum approximations to **I**. On the axes below sketch a graph of a function which has the property that on the given interval you can predict that the relative sizes of T, L, R, M and I will be: $R < T < I < M < L$

$R < I < L$ means f is decreasing



$T < I < M$ means f is concave down

Write several sentences below explaining how you know that the graph drawn above will indeed result in the relative sizes of T, L, R, M and I.

The curve must be decreasing and concave down.
Concave down means that Midpoint is an overestimate & Trapezoid is underestimate.
decreasing curve means Left is overestimate and Right is underestimate.

1998 EXAM 2

4. (20 points) Consider the integral $I_p = \int_1^\infty x^p \ln(x)\, dx$. We want to develop a rule for what values of p will I_p converge. Remember, p can be any real number. You should be able to find I_p regardless of what p actually is.

a. Apply the method of integration by substitution to I_p with $u = \ln(x)$. If we choose $u = \ln(x)$ this implies that $e^u = x$. Show that the integral I_p can be written completely in u-space as $\int_0^\infty u\, e^{(1+p)u}\, du$

TYPO!

$$I_p = \int_0^\infty (e^u)^p u \cdot x\, du = \int_0^\infty e^{up} \cdot e^u \cdot u\, du$$

$u = \ln x \Leftrightarrow x = e^u$ $\quad$ $x = 1, u = \ln 1 = 0$

$du = \frac{1}{x} dx$ $\quad$ $x \to \infty, u \to \infty$

$$I_p = \int_0^\infty e^{u(p+1)} u\, du$$

b. Evaluate the integral from part (a) to determine for what values of p I_p converges and for what values of p it diverges.

$$\int e^{u(p+1)} u\, du = \frac{u e^{u(p+1)}}{p+1} - \int \frac{e^{u(p+1)}}{p+1} du$$

$f' = e^{u(p+1)}$ $\quad$ $f = \frac{e^{u(p+1)}}{p+1}$

$g = u$ $\quad$ $g' = 1$

$$= \frac{u e^{u(p+1)}}{p+1} - \frac{1}{(p+1)^2} e^{u(p+1)}$$

$$\int_0^\infty e^{u(p+1)} u\, du = \lim_{b \to \infty} \frac{b e^{b(p+1)}}{p+1} - \frac{1}{(p+1)^2} e^{b(p+1)} = \begin{cases} 0 & p+1 < 0 \\ \infty & p+1 \geq 0 \end{cases}$$

$\lim_{b \to \infty} e^{b(p+1)} = \infty, p+1 > 0$; $0, p+1 < 0$

I_p: CON $p < -1$; DIV $p \geq -1$