Math 120 Spring 2014 — Exam #3

Name: BUCKMIRE

Thursday, April 17, 2014

Section 1:30pm | 3:00pm: (CIRCLE ONE)

Prof. Ron Buckmire

1. There are four (4) questions on this exam distributed on six (6) pages. (There is a 5 point BONUS question on the 7th page.) Each one involves various combination of analytic, verbal, computational and creative skills. Read and answer each question carefully and fully. **Your answers should be clearly communicated to the reader.**

3. Partial credit will be given, but only if I can see the correct parts of your solution method. Feel free to indicate what solution methods and concepts you are applying to each problem. In other words, **show all of your work.**

4. Recall the rules set out on the exam regulation handout. Only your "blue notes" and a writing implement are allowed. Your blue notes must be handed in with (and then stapled to) your exam. Before you are finished **please sign the pledge below.**

5. Take a deep breath, relax and enjoy yourself... I encourage questions!

Pledge: I, ______________________, pledge my honor as a human being and Occidental student, that I have followed all the rules above to the letter and in spirit.

No.	Score	Maximum
1		30
2		26
3		24
4		20
BONUS		5
Total		**100**

1. (*30 points*) **Convergence Tests for Infinite Series.** ANALYTIC, COMPUTATIONAL. Use an appropriate test for convergence or divergence to determine if each of the following series converges or diverges. After you have finished your application of the test, write the name of the test you uses and whether the series converges or diverges. For example, "CONVERGES by Foo Test."

a. (*5 points*) $\sum_{k=1}^{\infty} \ln(k)$ $\boxed{\text{DIVERGES by the } n^{th} \text{ term test}}$

$\lim_{k \to \infty} \ln k = \infty \neq 0$

b. (*5 points*) $\sum_{k=1}^{\infty} \frac{\ln(k)}{k}$ $\boxed{\text{DIVERGES by the Integral Test}}$

$$\int_1^{\infty} \frac{\ln k}{k}\,dk = \lim_{b \to \infty} \int_1^b \ln k \, d(\ln k) = \lim_{b \to \infty} \int_1^{\ln b} u\,du = \lim_{b \to \infty} \frac{u^2}{2}\Big|_1^{\ln b}$$

$$= \lim_{b \to \infty} \frac{(\ln b)^2}{2} - \frac{1}{2}$$

$$= \infty$$

c. (*5 points*) $\sum_{k=1}^{\infty} \frac{\ln(k)}{k^3}$ $\boxed{\text{CONVERGES by COMPARISON Test}}$

$\ln k < k$ for $k > 1$

$\frac{\ln k}{k^3} < \frac{k}{k^3}$

$\frac{\ln k}{k^3} < \frac{1}{k^2}$

$\sum \frac{1}{k^2}$ CONVERGES by p-series test

d. (*5 points*) $\sum_{k=1}^{\infty}(-1)^k \frac{k^2}{k^2+2k+1}$ $\boxed{\text{DIVERGES by Alt. Series Test}}$

$$\lim_{K\to\infty} \frac{K^2}{K^2+2K+1} = 1 \text{ (L'Hopital's Rule)}$$

e. (*5 points*) $\sum_{k=1}^{\infty} \frac{1}{k^2+2k+1}$ $\boxed{\text{CONVERGES by Comparison (or Integral)}}$

$$\sum \frac{1}{K^2+2K+1} = \sum_{k=1}^{\infty} \frac{1}{(K+1)^2} < \sum_{k=1}^{\infty} \frac{1}{K^2}$$

since $K+1 > K$

$(K+1)^2 > K^2$

$\frac{1}{(K+1)^2} < \frac{1}{K^2}$

$$\int_1^{\infty} \frac{1}{(K+1)^2}\,dK = \left.\frac{-1}{K+1}\right|_1^{\infty} = 0 - \frac{-1}{2} = \frac{1}{2}$$

f. (*5 points*) $\sum_{k=1}^{\infty}(-1)^k 2^{-k}$ $\boxed{\text{CONVERGES by geometric series test}}$

$$\sum_{K=1}^{\infty} \left(-\frac{1}{2}\right)^K$$

2. (*26 points*) ANALYTIC, VERBAL, CREATIVE. **Sequences, Series, Convergence.** Some students are discussing calculus and you overhear their conversation.

Sydney: The n-th term test is the best test to use for proving an infinite series converges! I just proved that the alternating harmonic series $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}$ diverges because I know that $\lim_{k\to\infty}\frac{1}{k}=0$.

Madison: I think that since it's an alternating series we should use the alternating series test. In addition to showing that the terms of the alternating series become zero in the limit (like you did), you also need to show that they diminish in magnitude. We know this is true also since when $k > 1$ we know $k+1$ is greater than k which means that $\frac{1}{k+1}$ is always less than $\frac{1}{k}$. So we have shown that the alternating harmonic series converges by the alternating series test.

Terry: Actually, we could have just used the theorem that absolute divergence implies divergence. Since we know the harmonic series $\sum_{k=1}^{\infty}\frac{1}{k}$ diverges, and we know the absolute version of the alternating harmonic series **is** the harmonic series. Thus since we know the harmonic series diverges the alterating harmonic series diverges as well.

The Student With The Best Understanding Of Calculus Concepts Is: MADISON

For *6 points* **write down the name of the student whose dialogue displays the best understanding of concepts in Calculus in the box.** In a short essay, comment on the understanding of calculus displayed by all the students. In clear, legible sentences identify any correct and incorrect mathematical statements made by any of the students. If a statement is incorrect explain why. **You must be careful not to make any incorrect statements yourself in your explanation.** PROOFREAD YOUR ANSWER.

Sydney doesn't realize the nth term test is only used to prove divergence, NOT convergence. Then Sydney uses the result that $\lim a_k = 0$ to prove convergence when we know there are plenty of series for which that is true but they diverge. Sydney doesn't know alt. harmonic series converges.

Madison applies the Absolute Ratio Test correctly.

Madison also correctly show that $\frac{1}{k}$ is a decreasing sequence by showing $\frac{1}{k+1} < \frac{1}{k}$ for all $k > 1$.

Terry makes up a theorem. Absolute convergence implies convergence. The same is <u>NOT</u> true about divergence.

Terry does know that harmonic series diverges.

3. (*24 points*) ANALYTIC, VERBAL, COMPUTATIONAL, CREATIVE. **Limits, Sequences, L'Hopital's Rule.** In each part of this question, you need to think up an example of a function whose limit has the required properties. In the first two parts of this problem, your choice **MUST INCLUDE THE EXPONENTIAL FUNCTION** $e^{\square}$ **SOMEWHERE (though it may also include other functions). You must then ALSO show or prove that the function you write down produces the correct, required limit.**

   a. (*9 points*) Choose a function $a(k)$ which, as $k \to \infty$ is **INFINITY**. In other words, find an $a(k)$ so that $\lim_{k\to\infty} a(k) = \infty$. **You must include the exponential function.**

$$\lim_{k\to\infty} e^k = \infty$$

any of these will work

$$\left\{ \begin{array}{l} a_k = e^k \\ a_k = e^{k^2} \\ a_k = e^1 + k \end{array} \right.$$

   b. (*9 points*) Choose a function $b(k)$ which, as $k \to \infty$ appears to have an **INDETERMINATE** value, but on further evaluation using L'Hopital's Rule is equal to **ZERO**. In other words, find an $b(k)$ so that $\lim_{k\to\infty} b(k) = \frac{\infty}{\infty} \overset{L'H}{=} 0$. **You must include the exponential function.**

$$b_k = \frac{e^k + 1}{e^{k^2} + 1}$$

$$\lim_{k\to\infty} \frac{\ln k}{e^k} = \frac{\infty}{\infty} \overset{L'H}{=} \lim_{k\to\infty} \frac{\frac{1}{k}}{e^k} = \frac{0}{\infty} = 0$$

$$b_k = \frac{k^p}{e^k} \quad (p > 0)$$

$$b_k = \frac{\ln k}{e^k}$$

   c. (*6 points*) Think of the $a(k)$ and $b(k)$ that you chose in part (a) and (b) as sequences. Discuss whether either of these sequences would converge.

As sequences, only b_k would converge, since $\lim_{k\to\infty} b_k = 0$

a_k diverges, since $\lim_{k\to\infty} a_k = \infty$

4. (*20 points*) **ANALYTIC, COMPUTATIONAL, VISUAL. Power Series, Interval of Convergence, Radius of Convergence**
Consider the mystery function $M(x)$ is represented by the power series

$$\sum_{k=1}^{\infty} \frac{2^k}{k} x^k$$

a. (*10 points*) Show that the radius of convergence of this series is $\frac{1}{2}$.

$$a_K = \frac{2^K x^K}{K}$$

$$a_{K+1} = \frac{2^{K+1} x^{K+1}}{K+1}$$

$$\frac{1}{R} = \lim_{K\to\infty} \left| \frac{\frac{2^{K+1}}{K+1}}{\frac{2^K}{K}} \right| = \lim_{K\to\infty} \frac{2^K \cdot 2 \cdot K}{2^K \cdot K+1} = \lim_{K\to\infty} \frac{2K}{K+1} = 2$$

$$\frac{1}{R} = 2 \Rightarrow R = \frac{1}{2}$$

Using Abs. Rat. Test

$$\left|\frac{a_{K+1}}{a_K}\right| = \left| \frac{\frac{2^{K+1} x^{K+1}}{K+1}}{\frac{2^K x^K}{K}} \right| = \left| \frac{2^{K+1}}{2^K} \cdot \frac{x^{K+1}}{x^K} \frac{K}{K+1} \right| = |x| \frac{2K}{K+1}$$

$$L = \lim_{K\to\infty} |x| \frac{2K}{K+1} = |x| \cdot 2 < 1$$

$$|x| < \frac{1}{2} = R$$

b. (*10 points*) For what values of x will the power series converge to $M(x)$? (Write down an inequality involving x or sketch a diagram indicating these values.)

Check $x = \frac{1}{2}$

$$\sum_{K=1}^{\infty} \frac{2^K \cdot \left(\frac{1}{2}\right)^K}{K} = \sum_{K=1}^{\infty} \frac{1}{K}$$

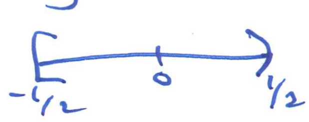
DIVERGES because Harmonic Series diverges!

Check $x = -\frac{1}{2}$

$$\sum_{K=1}^{\infty} \frac{2^K \cdot \left(\frac{-1}{2}\right)^K}{K} = \sum_{K=1}^{\infty} \frac{(-1)^K}{K}$$

CONVERGES because Alternating Harmonic series converges

Interval of convergence is $\left[-\frac{1}{2}, \frac{1}{2}\right)$ or $-\frac{1}{2} \le x < \frac{1}{2}$



BONUS (*5 points*) Recall from Question 4, the power series representation of a mystery function $M(x)$ is

$$\sum_{k=1}^{\infty} \frac{2^k}{k} x^k = 2x + \frac{(2x)^2}{2} + \frac{(2x)^3}{3} + \frac{(2x)^4}{4}$$

What is identity of the mystery function $M(x)$? PROVE YOUR ANSWER.

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \dots$$

Since $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots$

$$-\ln(1-2x) = (2x) + \frac{(2x)^2}{2} + \frac{(2x)^3}{3} + (2x)^4$$

$$M(x) = -\ln(1-2x)$$