Report on Exam 2

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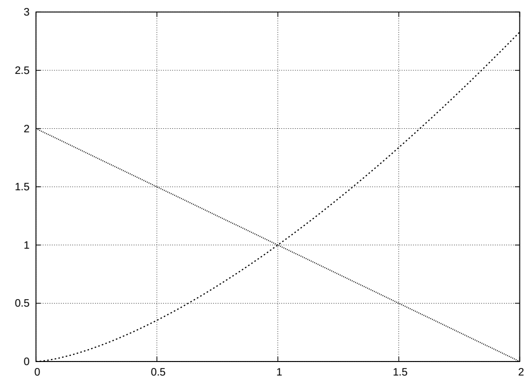
Point Distribution (N=21)

Range	92.5+	90+	87.5+	82.5+	80+	77.5+	72.5+	70+	67.5+	62.5+	60+	60-
Grade	A	A-	B+	B	B-	C+	C	C-	D+	D	D-	F
Frequency	5	2	1	4	1	1	2	2	1	0	0	2

Summary The results on Exam 2 were almost as good as on Exam 1 with an identical 8 of 21 students getting an A again, however this time there were two students getting an F. The median score was higher this time (84) but the mean was lower (82). The high score was 99. Congratulations to everyone!



#1 ANALYTIC, VISUAL, COMPUTATUTIONAL. **Area between curves, Curve length**. The first thing to do is draw a picture on the given graph of what region A is. It's the area that is touching the y-axis at the left and then bounded below by $y = x^{3/2}$ and above by $y = 2 - x$.

a. To find the area of the region you can just use the $\int y_T - y_B dx$ formula and integrate vertical slices (i.e. it's a dx integral) from $x = 0$ to $x = 1$ (which you can see from the gtaph is where the two curves intersect). So, $A = \int_0^1 2 - x - x^{3/2} dx$.

$$
\begin{aligned}
A = \int_0^1 2 - x - x^{3/2} dx &= 2x - \frac{x^2}{2} - \frac{2}{5}x^{5/2}\Big|_0^1 \\
&= 2 - \frac{1}{2} - \frac{2}{5} \\
&= \frac{20 - 5 - 4}{10} \\
A &= \frac{11}{10}.
\end{aligned}
$$

(b) To find the length of the perimieter of region A you have a vertical line segment aloing the y-axis (2 units), and the curved segment along $y = x^{3/2}$ and then an angled line segment from (1,1) to (0,2). Using Pythagoras Theorem (or distance formula) lets you know that segment is $\sqrt{2}$. For the curved segment you can use the arclength formula $\int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$ where since $y = x^{3/2}$ you know $\frac{dy}{dx} = \frac{3}{2}x^{1/2}$ and when you square that it becomes $\frac{9x}{4}$ so the entire length of the perimieter is $\sqrt{2} + 2 + \int_0^1 \sqrt{1 + \frac{9x}{4}} dx$.

#2 ANALYTIC, VISUAL, COMPUTATIONAL. **Volume of solids of revolution**. **(a)** When you rotate region A from problem 1 around the y-axis you can use either disks or shells for this problem. For disks the integral looks like $\pi\int_0^2 x^2 dy$ and becomes $\int_0^1 \pi(y^{2/3})^2 dy + \int_1^2 \pi(2-y)^2 dy$. Using shells the integral looks like $\int_0^1 2\pi x f(x)dx$ which becomes $\int_0^1 2\pi x(2-x-x^{3/2})dx = 2\pi\int_0^1 2x - x^2 - x^{5/2}dx = x^2 - \frac{x^3}{3} - \frac{2}{7}x^{7/2}\Big|_0^1$. This volume is $\frac{16\pi}{21}$.

(b) When you rotate region A from problem 1 around the x-axis you can use either washers or shells. For washers the integral looks like $\int_0^1 \pi y_{top}^2 - y_{bottom}^2 dx$ which becomes $\int_0^1 \pi(2-x)^2 - (x^{3/2})^3\, dx = \int_0^1 4 - 4x + x^2 - x^3\, dx = \pi\,(4x - 2x^2 + x^2 - \frac{x^4}{4})\Big|_0^1$. If you want to use shells the integral looks like $\int_0^2 2\pi y g(y)\, dy$. Specifically, it becomes $2\pi\int_0^1 y\cdot y^{2/3}\, dy + 2\pi\int_1^2 y\cdot(2-y)\, dy = 2\pi\int_0^1 y^{5/3}\, dy + 2\pi\int_1^2 2y - y^2\, dy$. This volume is $\frac{25\pi}{12}$.

#3 ANALYTIC, COMPUTATIONAL. **Differential equations, partial fractions. (a)** You can use separation of variables to solve $\frac{dw}{dt} = w^2 t, w(0) = -2$.

$$\begin{aligned}
\frac{dw}{dt} &= w^2 t \\
\frac{dw}{w^2} &= t dt \\
\int w^{-2} dw &= \int t\, dt \\
-\frac{1}{w} &= \frac{t^2}{2} + C \\
-\frac{1}{-2} &= \frac{0^2}{2} + C \qquad \text{(Using the intial condition that when } t = 0 \text{ and } w = -2) \\
\frac{1}{2} &= C \\
-\frac{1}{w} &= \frac{t^2}{2} + \frac{1}{2} \\
\frac{1}{w} &= -\frac{t^2}{2} - \frac{1}{2} \\
&= -\frac{(t^2+1)}{2} \\
w &= \frac{-2}{t^2+1}
\end{aligned}$$

To check your answer you have to check both the initial condition and differential equation. In other words, when $t = 0$ does $w = -2$? $w(0) = \frac{-2}{0^2+1} = -2$. **Yes!** And check the DE, does $w'(t)$ equal to $w^2 t$? $w'(t) = 2(t^2+1)^{-2}\cdot(2t) = t\cdot\frac{4}{(t^2+1)^2} = tw^2$. **Yes!** **(b)** You can use partial fractions to split $\frac{y+3}{(y-1)(y+1)} = \frac{A}{y-1} + \frac{B}{y+1}$. This turns into the equation $y+3 = A\cdot(y+1) + B\cdot(y-1)$. Letting $y = 1$ gives the equation $4 = 2\cdot A$ so $A = 2$ and letting $y = -1$ produces $2 = B\cdot(-2)$ so $B = -1$. Therefore $\int_2^3 \frac{y+3}{(y-1)(y+1)}dy = \int_2^3 \frac{2}{y-1} + \frac{-1}{y+1}dy = 2\ln|y-1| - \ln|y+1||_2^3 = (2\ln 2 - \ln 4) - (2\ln 1 - \ln 3) = \ln(3)$.

#4 ANALYTIC, VERBAL, COMPUTATATIONAL. **Improper Integrals, Numerical Integration.**

(a) FALSE "If a function $f(x) > 0$ for all $x > 1$ and $\lim_{x\to\infty} f(x) = 0$, then $\int_1^\infty f(x)\,dx$ must **converge**." Because this statement has to always be true it is claiming that EVERY function which has the property $f(x) > 0$ for all $x > 1$ and $\lim_{x\to\infty} f(x) = 0$ will produce a convergent improper integral of the first kind $\int_1^\infty f(x)\,dx$. But there are a lot of divergent integrals, so all one has to do is find one that has an integrand with those properties to provide a counter-example. The most immediate one is $f(x) = \dfrac{1}{x}$. This function has the required properties that $\lim_{x\to\infty} \dfrac{1}{x} = 0$ and $\frac{1}{x} > 0$ for all $x > 1$ but we also know that $\int_1^\infty \dfrac{1}{x}\,dx$ DIVERGES so the statement is not true for every function.

(b) TRUE. "If a function $f(x) > \dfrac{1}{x} > 0$ for all $x > 1$, then $\int_1^\infty f(x)\,dx$ must **diverge**." This is basically a statement of the comparison theorem where the function being compared to is $\frac{1}{x}$. Since we know that when $\frac{1}{x}$ is the integrand of an improper integral of the first kind the integral DIVERGES, and we know that $f(x) > \frac{1}{x} > 0$ then we know that $\int_1^\infty f(x)\,dx > \int_1^\infty \dfrac{1}{x}\,dx$ so our new integral must be **divergent** as well.

(c) FALSE. "If you use the Midpoint Riemann sums to approximate a definite integral and you know that this estimate is an over-estimate, then it is also true that using Right-hand Riemann sums will give you an under-estimate." You know that when $f(x)$ is continuous and monotonic on an interval that $L < I < R$ if f is an increasing function (i.e. $f' > 0$) and $L > I > R$ if f is a decreasing function, where I is the exact value of the integral and L is the left-hand Riemann Sum approximation and R is the Right-hand Riemann Sum approximation. You should also know that when f is continuous and concave up (i.e. $f'' > 0$) that $M < I < T$ where M is the Midpoint Riemann Sun approximation and T is average of the Left and Right Riemann sums (also known as the Trapezoid approximation). If one has a continuous function that is concave down and increasing, Midpoint will be an over-estimate and Right-hand will be an over-estimate, so the scenario in the given statement (i.e. M over-estimate and R under-estimate) is not always true. There is no related "over-under" relationship between M and R, as there is between M and T and between R and L