Math 120 Spring 2014 — Exam #2

Name: BUCKMIRE

Thursday, March 20, 2014

Section 1:30pm | 3:00pm: (CIRCLE ONE)

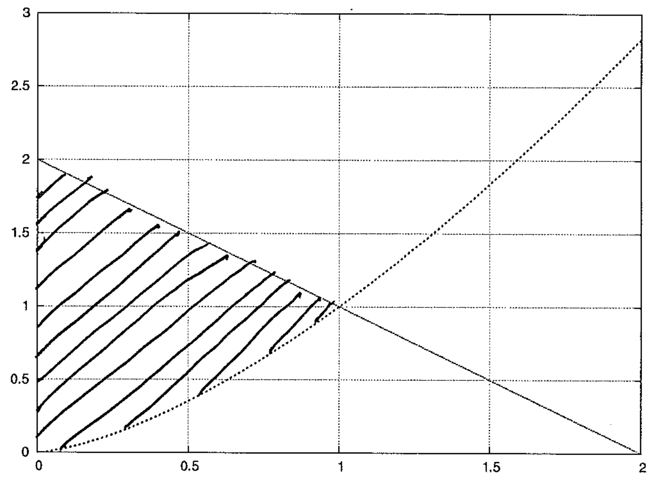
Prof. Ron Buckmire

1. There are four (4) questions on this exam distributed on five (5) pages. Each one involves varius combination of analytic, verbal, computational and visual skills. Read and answer each question carefully and fully. **Your answers should be clearly communicated to the reader.**
3. Partial credit will be given, but only if I can see the correct parts of your solution method. Feel free to indicate what solution methods and concepts you are applying to each problem. In other words, **show all of your work.**
4. Recall the rules set out on the exam regulation handout. Only your "blue notes" and a writing implement are allowed. Your blue notes must be handed in with (and then stapled to) your exam. Before you are finished **please sign the pledge below.**
5. Take a deep breath, relax and enjoy yourself... I encourage questions!

Pledge: I, ____________________, pledge my honor as a human being and Occidental student, that I have followed all the rules above to the letter and in spirit.

No.	Score	Maximum
1		20
2		24
3		26
4		30
Total		**100**

1. (*20 points*) ANALYTIC, VISUAL, COMPUTATUTIONAL. **Area between curves, Curve length.** Consider the region A bounded by the y-axis, $y = x^{3/2}$ and $y = 2 - x$.



a. (*10 points*) Find the area of the region A (which is between $y = x^{3/2}$, $y = 2 - x$ and the y-axis).

$$A = \int_0^1 2 - x - x^{3/2}\, dx$$

$$= \left(2x - \frac{x^2}{2} - \frac{2x^{5/2}}{5}\right)\Big|_0^1$$

$$= 2 - \frac{1}{2} - \frac{2}{5} =$$

b. (*10 points*) Find the perimeter (length of the boundary) of the region A. (You may leave a definite integral in your answer.)

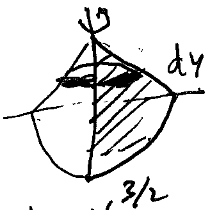
$$L = 2 + \sqrt{2} + \int_0^1 \sqrt{1 + [(x^{3/2})']^2}\, dx$$

$$= 2 + \sqrt{2} + \int_0^1 \sqrt{1 + \left(\frac{3}{2}x^{1/2}\right)^2}\, dx$$

$$= 2 + \sqrt{2} + \int_0^1 \sqrt{1 + \frac{9x}{4}}\, dx$$

2. (*24 points*) ANALYTIC, VISUAL, COMPUTATIONAL. **Volume of solids of revolution**. Consider again (from **Problem 1**) the region A bounded by the y-axis, $y = x^{3/2}$ and $y = 2 - x$.

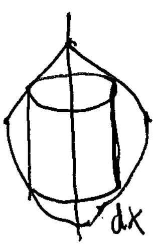
a. (*12 points*) Show that the volume of the solid formed by rotating the region A around the y-axis is $\frac{16\pi}{21}$. (State the method you use: shells, washers or disks. The bulk of your credit will be for setting up the correct definite integral, not in evaluating it.)



$y = x^{3/2}$

$y^2 = x^3$

$y^{2/3} = x$



DISKS

$$V = \int_0^2 \pi x^2\, dy$$

$$= \int_0^1 \pi (y^{2/3})^2\, dy + \int_1^2 \pi (2-y)^2\, dy$$

$$= \pi \int_0^1 y^{4/3}\, dy + \pi \int_1^2 4 - 4y + y^2\, dy$$

$$= \pi\, \frac{3}{7} y^{7/3}\Big|_0^1 + \pi \left(4y - 2y^2 + \frac{y^3}{3}\right)\Big|_1^2$$

$$= \pi \frac{3}{7} + \pi \left(8 - 8 + \frac{8}{3}\right) - \left(4 - 2 + \frac{1}{3}\right)$$

$$= \frac{3\pi}{7} + \pi\left(\frac{8}{3} - \frac{7}{3}\right) = \frac{9\pi}{21} + \frac{7\pi}{21} = \frac{16\pi}{21}$$

Shells

$$V = \int_0^1 2\pi x f(x)\, dx$$

$$= \int_0^1 2\pi x \left(2 - x - \sqrt{x^3}\right) dx$$

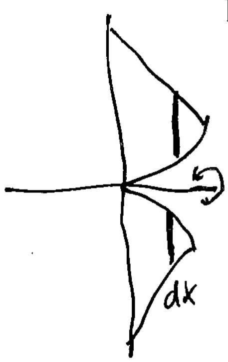
$$= 2\pi \int_0^1 2x - x^2 - x^{5/2}\, dx$$

$$= 2\pi \left(x^2 - \frac{x^3}{3} - \frac{2x^{7/2}}{7}\right)\Big|_0^1$$

$$= 2\pi \left(1 - \frac{1}{3} - \frac{2}{7}\right)$$

$$= 2\pi \left(\frac{21 - 7 - 6}{21}\right) = 2\pi \cdot \frac{8}{21} = \frac{16\pi}{21}$$

b. (*12 points*) Show that the volume of the solid formed by rotating the region A around the x-axis is $\frac{25\pi}{12}$. (State the method you use: shells, washers or disks. The bulk of your credit will be for setting up the correct definite integral, not in evaluating it.)



Washer

$$V = \int_0^1 \pi y_T^2 - \pi y_B^2\, dx$$

$$= \int_0^1 \pi (2-x)^2 - \pi \left(x^{3/2}\right)^2 dx$$

$$V = \pi \int_0^1 4 - 4x + x^2 - x^3\, dx$$

$$= \pi \left(4x - 2x^2 + \frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1$$

$$= \pi \left(4 - 2 + \frac{1}{3} - \frac{1}{4}\right)$$

$$= \pi \left(\frac{24 + 4 - 3}{12}\right) = \left(\frac{28 - 3}{12}\right)\pi$$

$$= \frac{25\pi}{12}$$

Shells

$$V = \int_0^2 2\pi y\, g(y)\, dy$$

$$= 2\pi \int_0^1 y \cdot y^{2/3}\, dy + 2\pi \int_1^2 y(2-y)\, dy$$

$$= 2\pi \int_0^1 y^{5/3}\, dy + 2\pi \int_1^2 2y - y^2\, dy$$

$$= 2\pi \left(\frac{3y^{8/3}}{8}\right)\Big|_0^1 + 2\pi \left(y^2 - \frac{y^3}{3}\right)\Big|_1^2$$

$$= 2\pi \cdot \frac{3}{8} + 2\pi \left[\left(4 - \frac{8}{3}\right) - \left(1 - \frac{1}{3}\right)\right]$$

$$= \frac{3\pi}{4} + 2\pi \left[\frac{4}{3} - \frac{2}{3}\right]$$

$$= \frac{3\pi}{4} + \frac{4\pi}{3} = \frac{9\pi + 16\pi}{12} = \frac{25\pi}{12}$$

3. (*26 points*) **ANALYTIC, COMPUTATIONAL. Differential equations, partial fractions.**

a. (*14 points*) Find the function $w(t)$ which solves the initial value problem

$$\frac{dw}{dt} = tw^2, \qquad w(0) = -2$$

Check your answer!

$$\frac{dw}{w^2} = t\,dt$$

$$\int \frac{dw}{w^2} = \int t\,dt$$

$$-\frac{1}{w} = \frac{t^2}{2} + C$$

$$-\frac{1}{-2} = C \Rightarrow C = \frac{1}{2}$$

$$-\frac{1}{w} = \frac{t^2}{2} + \frac{1}{2} = \frac{t^2+1}{2}$$

$$\frac{1}{w} = -\left(\frac{t^2+1}{2}\right)$$

$$\boxed{w = -\frac{2}{t^2+1}}$$

Check IC

$t = 0, w = -2$

$w = -\frac{2}{0^2+1} = -2$ ✓

Check DE

$$w' = -2 \cdot \frac{-1}{(t^2+1)^2} \cdot 2t$$

$$= \frac{4}{(t^2+1)^2} \cdot t$$

$$= w^2 \cdot t \checkmark$$

b. (*12 points*) Show that $\displaystyle\int_2^3 \frac{y+3}{(y-1)(y+1)}\,dy = \ln(3)$.

$$\frac{y+3}{(y-1)(y+1)} = \frac{A}{y-1} + \frac{B}{y+1} = \frac{2}{y-1} + \frac{-1}{y+1}$$

$$y+3 = A(y+1) + B(y-1)$$

$y=1$ $\quad 4 = 2 \cdot A \Rightarrow A = 2$

$y=-1$ $\quad 2 = B(-2) \Rightarrow B = -1$

$$\int_2^3 \frac{y+3}{(y-1)(y+1)} = \{2\ln|y-1| - \ln|y+1|\}\Big|_2^3$$

$$= \{2\ln 2 - \ln 4\} - \{2\ln 1 - \ln 3\}$$

$$= \ln 3$$

4. (*30 points*) ANALYTIC, VERBAL, COMPUTATATIONAL. **Improper Integrals, Numerical Integration.** Consider the following statements and write TRUE or FALSE in the adjacent box. **To be TRUE, the statement must ALWAYS be true.** If you think the statement is FALSE, give an example which contradicts the given statement. If you think the statement is TRUE, provide work or calculations which support this conclusion. You will receive **2 points** for your correct choice of TRUE/FALSE and **8 points** for your explanation or counterexample.

a. (*10 points*) **TRUE or FALSE?** "If a function $f(x) > 0$ for all $x > 1$ and $\lim_{x\to\infty} f(x) = 0$, then $\int_1^\infty f(x)\,dx$ must **converge.**"

$\boxed{\text{FALSE}}$

$f(x) = \frac{1}{x} > 0$ for all $x > 1$

and $\lim_{x\to\infty} \frac{1}{x} = 0$ but $\int_1^\infty \frac{1}{x}dx$

DIVERGES

b. (*10 points*) **TRUE or FALSE?** "If a function $f(x) > \frac{1}{x} > 0$ for all $x > 1$, then $\int_1^\infty f(x)\,dx$ must **diverge.**"

$\boxed{\text{TRUE}}$

By the Comparison Theorem

since $\int_1^\infty \frac{1}{x}dx$ DIVERGES, if for $x>1$

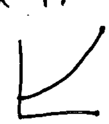
$f(x) > \frac{1}{x}$ then $\int_1^\infty f(x)dx$ must also diverge

c. (*10 points*) **TRUE or FALSE?** "If you use the Midpoint Riemann sums to approximate a definite integral and you know that this estimate is an over-estimate, then it is also true that using Right-hand Riemann sums will give you an under-estimate."

$\boxed{\text{FALSE}}$

Midpoint Riemann sums are over-estimate means that $f(x)$ is concave up.

If could be concave up and INCREASING



or

DECREASING

No relationship between midpoint and Righthand Riemann sums

$M \propto f''$ or concavity

$L, R \propto f'$ or increase/decrease