Taylor Series for

$f(x) = e^{-x^3}$ about the point $x = -1.$

**You have to take derivatives**

$$T_n(x) = f(-1) + f'(-1)(x+1) + \frac{f''(-1)(x+1)^2}{2!} + \ldots$$

$$f(x) = e^{-x^3} \quad f(-1) = e^{-(-1)^3} = e^1$$

$$f'(x) = -3x^2 \quad f'(-1) = e^1 \cdot -3(-1)^2 = -3e$$

$$f''(x) = e^{-x^3}(-3x^2)^2 + e^{-x^3}(-6x) \quad f''(-1) = e^{-1} \cdot 9 + e^6$$

$$T_n(x) = e^{-3e(x+1)} + \frac{e^6}{15}(x+1)^2 + \ldots$$

The question probably should have been

Find Taylor Series for $f(x) = e^{-x^3}$ about $x = 0$.

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots + \frac{x^k}{k!}$$

$$e^{-x^3} = 1 - x^3 + \frac{(-x^3)^2}{2!} + \frac{(-x^3)^3}{3!} + \ldots + \frac{(-x^3)^k}{k!}$$

$$= 1 - x^3 + \frac{x^6}{2!} - \frac{x^9}{3!} + \frac{x^{12}}{4!} + \ldots$$
15. Find radius & interval of convergence

\[ \sum_{k=0}^{\infty} (-4)^k x^{2k+1} \]

**Absolute Ratio Test**

\[
L = \lim_{k \to \infty} \left| \frac{(-4)^{k+1} x^{2k+3}}{(-4)^k x^{2k+1}} \right| = \lim_{k \to \infty} \left| (-4) \frac{x^{2k+3}}{x^{2k+1}} \right| = \lim_{k \to \infty} \left| -4x^2 \right| = 4x^2
\]

\[ L = 4x^2 < 1 \]

For convergence \( L < 1 \) so \( 4x^2 < 1 = \frac{1}{2} \) and \( -\frac{1}{2} < x < \frac{1}{2} \)

So the radius of convergence is \( \frac{1}{2} \)

We need to check the endpoints to find the interval of convergence

At \( x = \frac{1}{2} \)

\[ \sum_{k=0}^{\infty} (-4)^k \left( \frac{1}{2} \right)^{2k+1} = \sum_{k=0}^{\infty} (-4)^k \left( \frac{1}{2} \right)^{2k+1} = \sum_{k=0}^{\infty} \left( -1 \right)^k \left( 2 \right)^k \frac{1}{2^{2k+1}} \]

DIVERGES by n-th term test

At \( x = -\frac{1}{2} \)

\[ \sum_{k=0}^{\infty} (-4)^k \left( -\frac{1}{2} \right)^{2k+1} = \sum_{k=0}^{\infty} (-1)^k \left( -1 \right)^k \left( \frac{1}{2} \right)^{2k+1} = \sum_{k=0}^{\infty} (-1)^k \left( -1 \right)^{k+1} \frac{1}{2} \]

DIVERGES by n-th term test

\[ \lim_{k \to \infty} (-1)^k \frac{1}{2} = \text{DNE} \neq 0 \]
16. \( f(x) = \begin{cases} 0, & -\pi < x < -\frac{\pi}{2} \\ 1, & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < x < \pi \end{cases} \)

Sketch the graph of \( f(x) \) and find its period.

**Period is** \( 2\pi \)

\[ f(-x) = f(x) \]

Symmetric about y-axis so it's even.

\[ q_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 \, dx = \frac{1}{2\pi} \left[ x \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{1}{2} \]

Area under graph: \( \frac{\pi \cdot 1}{2\pi} = \frac{1}{2} \)

We know all \( b_k = 0 \) because the function is even so no SINE coefficients.

\[ a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) \, dx = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 \cdot \cos(kx) \, dx \]

\[ = \sin(kx) \bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \sin \left( \frac{k\pi}{2} \right) - \sin \left( -\frac{k\pi}{2} \right) = \frac{2 \sin \left( \frac{k\pi}{2} \right)}{k\pi} \]

\[ a_k = \frac{2 \sin \left( \frac{k\pi}{2} \right)}{k\pi} \]

\[ A_0 = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2 \sin(k\pi)}{k\pi} \cos(kx) = \frac{1}{2} + \frac{2}{\pi} \cos x + \frac{2}{3\pi} \cos 3x \]

\[ + \frac{2}{5\pi} \cos 5x + \ldots \]
\[ \int_0^{\pi/2} 2\pi x \cos x \, dx \leftarrow \text{This uses the method of shells.} \]

This is the volume formed by rotating the area bounded by \( \cos x \) between \( x = 0 \) and \( x = \pi/2 \) and the \( x \)-axis around the \( y \)-axis.

\[ \int_0^{\pi/2} \pi (2 - \sin x)^2 \, dx \leftarrow \]

This used method of washers.

This is the volume formed by rotating the \( y = \sin x \) curve between \( x = 0 \) and \( x = \pi/2 \) about the \( y = 2 \) line.

\[ \int_0^{\pi/2} \pi (6 - y) (4y - y^2) \, dy \]

This is the volume formed by rotating the area \( 4y - y^2 \) between \( y = 0 \) and \( y = 4 \) around the \( y = 6 \) axis.

\[ x = 4y \Rightarrow y = \frac{x}{4} \quad \text{The } y = 6 \text{ axis.} \]

\[ x = y^2 \quad \text{This is the same area as between } y = \sqrt{x} \text{ and } y = \frac{x}{4} \text{ on } x = 0 \text{ to } x = 16 \]
Stewart, pg 422, #14

(a) Area of $R = \int_0^1 2x - x^2 - x^3\,dx$

\[
= \left[ \left( \frac{2x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \right) \right]_0^1 = 1 - \frac{1}{3} - \frac{1}{4} = \frac{12}{12} - \frac{4}{12} - \frac{3}{12} = \frac{5}{12}
\]

(b) Rotate about the $x$-axis

\[V = \int_0^1 \pi (2x-x^2)^2 - (x^3)^2\,dx = \frac{2}{3} \pi \text{ can't use shells! bec cause } y = 2x - x^2 \text{ can't be inverted}
\]

Using Washers

\[= \pi \int_0^1 \left( 4x^2 - 4x^3 + x^4 - x^6 \right) - \pi \left( \frac{4x^2 - x^4 + 2 - x^2}{3} \right) \,dx = \pi \left[ \frac{x^3}{2} - \frac{x^4}{4} - \frac{x^4}{5} \right]_0^1 = \pi \left[ \frac{1}{2} + \frac{1}{12} - \frac{17}{7} \right] = \pi \left[ \frac{35 + 21 - 15}{105} \right] = \frac{41}{105} \pi
\]

(c) Rotate about the $y$-axis

\[V = \int_0^1 2\pi x (2x-x^2-x^3)\,dx
\]

Using Shells

\[= 2\pi \int_0^1 2x^2 - x^3 - x^4\,dx
\]

\[= 2\pi \left[ \frac{2x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} \right]_0^1 = 2\pi \left( \frac{40}{60} - \frac{15}{60} - \frac{12}{60} \right) = \frac{26\pi}{60} = \frac{13\pi}{30}
\]