2003 EXAM3

1. *(30 points total.)* **Convergence Tests.** Consider the infinite series $\sum_{k=0}^{\infty} e^k$.

Give THREE separate proofs (i.e. using convergence tests) to show that this infinite series DIVERGES.

a. Show that $\sum_{k=0}^{\infty} e^k$ diverges.

$\lim_{k\to\infty} e^k = \infty \neq 0$ So $\sum e^k$ diverges by n^{th} term test

b. Use a different test from (a) to again show that $\sum_{k=0}^{\infty} e^k$ diverges.

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{e^{k+1}}{e^k}\right| = \lim_{k\to\infty}|e| = e>1$$

Diverges by Absolute Ratio Test

c. Use a different test from (a) and (b) to again show that $\sum_{k=0}^{\infty} e^k$ diverges.

$$\lim_{k\to\infty}\sqrt[k]{|e^k|} = \lim_{k\to\infty}|e^k|^{1/k} = \lim_{k\to\infty} e = e>1$$

Diverges by root test

$\sum_{k=0}^{\infty} e^k = 1+e+e^2+\dots$ geometric series with $e>1$ implies divergence

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4. (20 points) Two students are discussing calculus and you overhear their conversation.

Sydney: The zero-limit test is the best test for infinite series! I just proved that the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ converges because I know $\lim_{k\to\infty} \frac{1}{k} = 0$

Madison: That's not right! You should use the comparison test. Show that $\frac{1}{k}$ is greater than 1 for all $k > 1$. Then since we know $\frac{1}{k}$ is positive for all $k > 1$ and since $\sum_{k=1}^{\infty} 1$ DIVERGES, this will prove that the harmonic series is greater than a divergent series, and thus also diverges.

Comment on the understanding of calculus displayed by the two students. In clear, legible sentences identify any correct and incorrect statements made by the students. If a statement is incorrect explain why. **You must be careful not to make any incorrect statements yourself in your explanation.** PROOFREAD YOUR ANSWER.

Sydney is wrong.
$\sum \frac{1}{K}$ is the harmonic series, which EVERYBODY knows diverges!
Just because $\lim_{K\to\infty} \frac{1}{K} = 0$ this does <u>not</u> mean $\sum_{K=1}^{\infty} \frac{1}{K}$ converges.

We know $\sum_{K=1}^{\infty} 1$ diverges because $\lim_{K\to\infty} 1 = 1 \neq 0$ (n^{th} term test).
Since $\frac{1}{K} < 1$ for all $K > 1$, then by comparison test $\sum \frac{1}{K}$ also diverges.

2001 EXAM 3

1. (30 points) Determine if the following infinite series converge or diverge. In each case, state which test you use and show how you apply the test.

a. $\sum_{k=2}^{\infty}(-1)^k \frac{1}{\ln(k)}$

(1) $\lim_{k \to \infty} \frac{1}{\ln k} = 0$ (2) $\frac{1}{\ln(k+1)} < \frac{1}{\ln k} \Rightarrow \ln k < \ln(k+1)$

so terms are decreasing

~~We know $k > \ln k$ for $k > 2$~~

~~so, we know $\frac{1}{k} < \frac{1}{\ln k}$~~

~~$\sum \frac{1}{\sqrt{k}}$ diverges (harmonic series) so $\sum \frac{1}{\ln k}$~~

~~diverges by the comparison test~~

Converges by Alternating Series Test

b. $\sum_{k=0}^{\infty} \sin(\frac{k\pi}{2}) = 0 + 1 + 0 + -1 + 0 + 1 + 0 - 1 + 0$

$\lim_{k \to \infty} \sin\left(\frac{k\pi}{2}\right) =$ Does Not Exist

The series diverges by nth term test.

c. $\sum_{k=1}^{\infty} \frac{k+1}{k}$

$a_k = \frac{k+1}{k} = 1 + \frac{1}{k}$

$\lim_{k \to \infty} 1 + \frac{1}{k} = 1 \neq 0 \Rightarrow$ DIVERGENCE by nth term test

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d. $\sum_{k=0}^{\infty} e^{-k} = 1 + \frac{1}{e} + \left(\frac{1}{e}\right)^2 + \cdots$

Geometric Series with $r = \frac{1}{e} < 1 \Rightarrow$ convergence

e. $\sum_{k=0}^{\infty} (-e)^k = 1 - e + e^2 - e^3 + \cdots$

$r = -e$

Geometric Series with $|-e| = e > 1 \Rightarrow$ DIVERGES

f. $\sum_{k=1}^{\infty} k^{-e}$

$$\int_1^{\infty} K^{-e}\,dK = \lim_{b\to\infty} \int_1^b \frac{1}{K^e}\,dK = \lim_{b\to\infty} \frac{1}{-e+1} \frac{1}{K^{e+1}} \Big|_1^b$$

$$= \frac{1}{1-e} \lim_{b\to\infty} \left(\frac{1}{b^{1+e}} - 1\right) = \frac{1}{1-e}(-1) = \frac{1}{e-1}$$

Converges by Integral Test

p-series with $p = e > 1 \Rightarrow$ CONVERGES

2000 EXAM3

2. (a) Find the Taylor series for the funtion $f(x) = \ln(x)$ centered at $a = 1$.

$f(x) = \ln x \quad f(a) = \ln 1 = 0$

$f'(x) = \frac{1}{x} \quad f'(a) = \frac{1}{1}$

$f''(x) = \frac{-1}{x^2} \quad f''(a) = \frac{-1}{1}$

$f'''(x) = \frac{2}{x^3} \quad f'''(a) = 2$

$f^{(iv)}(x) = -\frac{3 \cdot 2}{x^4} \quad f^{(iv)}(1) = -6 \quad f^{(v)}(x) = 24$

$f^{(k)}(1) = (-1)^{k+1}(k-1)!$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-1)^k = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(k-1)!\,(x-1)^k}{k!}$$

$$= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(k-1)!\,(x-1)^k}{k\,(k-1)!}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(x-1)^k$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + \dots$$

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(b) Find the interval of convergence for the Taylor Series you found in part (a).

$c_k = \frac{(-1)^{k+1}}{k}$

$c_{k+1} = \frac{(-1)^{k+2}}{k+1} = \frac{(-1)^k}{k+1}$

Interval $0 < x \le 2$

$$\frac{1}{R} = \lim_{k\to\infty} \left|\frac{c_{k+1}}{c_k}\right| = \lim_{k\to\infty} \frac{\frac{1}{k+1}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{k}{k+1} = 1$$

$R = 1$

At $x = 2$

$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ CONVERGES (alt. harmonic series)

At $x = 0$

$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}(-1)^k}{k} = \sum_{k=1}^{\infty} -\frac{1}{k}$ DIVERGES (harmonic series)

(c) What is the Power series for $g(x) = x\ln(x) - x$ at $a = 1$?

$$\ln(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \dots$$

$$\int \ln x \, dx = \int x - 1 - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \dots dx$$

$$x\ln x - x = \frac{(x-1)^2}{2} - \frac{1}{6}(x-1)^3 + \frac{1}{12}(x-1)^4 - \dots$$

1998 EXAM3

2. (30 points) Determine if the following infinite series converge or diverge. In each case, state which test you use and show how the test works.

a. $\sum_{k=0}^{\infty} k^{\pi}$ $= \sum_{K=0}^{\infty} \frac{1}{K^{-\pi}}$

p-series with $p=-\pi<1$
DIVERGES

b. $\sum_{k=0}^{\infty} \pi^{k}$

Geometric series with $\pi = r > 1 \Rightarrow$ DIVERGENCE

c. $\sum_{k=1}^{\infty} \frac{1}{k^{\pi}}$ $= \sum_{K=1}^{\infty} \frac{1}{K^{\pi}}$

$p=\pi>1$
p-series with $p=\pi>1$
CONVERGENCE

1998 EXAM3

d. $\sum_{k=0}^{\infty} k^{1/\pi}$ $= \sum_{K=0}^{\infty} K^{1/\pi}$

$\lim_{K\to\infty} K^{1/\pi} = \infty \neq 0$ DIVERGES by n^{th} term test

e. $\sum_{k=0}^{\infty} \left(\frac{1}{\pi}\right)^{k}$

Geometric series with $r=\frac{1}{\pi}<1 \Rightarrow$ convergent series

f. $\sum_{k=1}^{\infty} \frac{1}{k^{1/\pi}}$ $= \sum_{K=1}^{\infty} \frac{1}{K^{1/\pi}}$

$p=\frac{1}{\pi}<1$
DIVERGENT p-series