1. (30 points total.) Convergence Tests. Consider the infinite series $\sum_{k=0}^{\infty} e^k$.
Give THREE separate proofs (i.e. using convergence tests) to show that this infinite series DIVERGES.

a. Show that $\sum_{k=0}^{\infty} e^k$ diverges.

\[
\lim_{k \to \infty} e^k = \infty \neq 0 \quad \text{so} \quad \sum_{k=0}^{\infty} e^k \text{ diverges by n-th term test}
\]

b. Use a different test from (a) to again show that $\sum_{k=0}^{\infty} e^k$ diverges.

\[
\lim_{k \to \infty} \left( \frac{a_{k+1}}{a_k} \right) = \lim_{k \to \infty} \left( \frac{e^{k+1}}{e^k} \right) = \lim_{k \to \infty} e = e > 1
\]

Diverges by Absolute Ratio Test

c. Use a different test from (a) and (b) to again show that $\sum_{k=0}^{\infty} e^k$ diverges.

\[
\lim_{k \to \infty} \sqrt[k]{e^k} = \lim_{k \to \infty} (e^k)^{1/k} = \lim_{k \to \infty} e = e > 1
\]

Diverges by root test

$\sum_{k=0}^{\infty} e^k = 1 + e + e^2 + \ldots \quad \text{geometric series with } e > 1 \text{ implies divergence}$
4. (20 points) Two students are discussing calculus and you overhear their conversation.

Sydney: The zero-limit test is the best test for infinite series! I just proved that the harmonic series \( \sum_{k=1}^{\infty} \frac{1}{k} \) converges because I know \( \lim_{k \to \infty} \frac{1}{k} = 0 \).

Madison: That's not right! You should use the comparison test. Show that \( \frac{1}{k} \) is greater than 1 for all \( k > 1 \). Then since we know \( \frac{1}{k} \) is positive for all \( k > 1 \) and since \( \sum_{k=1}^{\infty} 1 \) DIVERGES, this will prove that the harmonic series is greater than a divergent series, and thus also diverges.

Comment on the understanding of calculus displayed by the two students. In clear, legible sentences identify any correct and incorrect statements made by the students. If a statement is incorrect explain why. You must be careful not to make any incorrect statements yourself in your explanation. PROOFREAD YOUR ANSWER.

Sydney is wrong.

\( \sum_{k=1}^{\infty} \frac{1}{k} \) is the harmonic series, which EVERYBODY knows diverges!

Just because \( \lim_{k \to \infty} \frac{1}{k} = 0 \) this does not mean \( \sum_{k=1}^{\infty} \frac{1}{k} \) converges.

We know \( \sum_{k=1}^{\infty} \frac{1}{k} \) diverges because \( \lim_{k \to \infty} \frac{1}{k} = 1 \neq 0 \) (n'th term test).

Since \( \frac{1}{k} < 1 \) for all \( k > 1 \), then by comparison test \( \sum_{k=1}^{\infty} \frac{1}{k} \) also diverges.
1. (30 points) Determine if the following infinite series converge or diverge. In each case, state which test you use and show how you apply the test.

a. \( \sum_{k=2}^{\infty} \frac{(-1)^k}{\ln(k)} \)

   (1) \( \lim_{k \to \infty} \frac{1}{\ln(k)} = 0 \)

   (2) \( \frac{1}{\ln(k+1)} < \frac{1}{\ln(k)} \) so terms are decreasing

   Converges by Alternating Series Test

   We know \( \sum \frac{1}{ln(k)} \) diverges, so diverges by comparison test

b. \( \sum_{k=0}^{\infty} \sin\left(\frac{k\pi}{2}\right) = 0 + 1 + 0 + -1 + 0 + 1 + 0 + -1 + \ldots \)

   \( \lim_{k \to \infty} \sin\left(\frac{k\pi}{2}\right) = \text{Does Not Exist} \)

   The series diverges by n-th term test.

c. \( \sum_{k=1}^{\infty} \frac{k+1}{k} \)

   \( a_k = \frac{k+1}{k} = 1 + \frac{1}{k} \)

   \( \lim_{k \to \infty} 1 + \frac{1}{k} = 1 + 0 \Rightarrow \text{Divergence by n-th term test} \)
d. \( \sum_{k=0}^{\infty} e^{-k} = 1 + \frac{1}{e} + \left(\frac{1}{e}\right)^2 + \ldots \)

Geometric series with \( r = \frac{1}{e} < 1 \Rightarrow \text{convergence} \)

e. \( \sum_{k=0}^{\infty} (-e)^k = 1 - e + e^2 - e^3 + \ldots \)

\( r = -e \)

Geometric Series with \( |r| = e > 1 \Rightarrow \text{DIVERGES} \)

\[
\int_{K}^{b} e^{-K} dK = \lim_{b \to \infty} \int_{K}^{b} e^{-K} dK = \lim_{b \to \infty} \left[ -\frac{1}{e} e^{-K} \right]_{K}^{b} \\
= \frac{1}{1 - e} \lim_{b \to \infty} \left( \frac{1}{b} - 1 \right) = \frac{1}{1 - e} \cdot (-1) = \frac{1}{e - 1}
\]

Converges by Integral Test

p-series with \( p = e > 1 \Rightarrow \text{CONVERGES} \)
(a) Find the area of the region where the graph you found in part (a)

\[ \int_{-1}^{1} \frac{x}{1 + x^4} \, dx = \frac{1}{2} \ln |1 + x^2| \bigg|_{-1}^{1} = 0 \]

(b) Find the length of the curve given by the function \( y = \arctan(x) \) from \( x = 0 \) to \( x = 1 \).

\[ L = \int_{0}^{1} \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx = \int_{0}^{1} \sqrt{1 + \frac{1}{1 + x^2}} \, dx = \int_{0}^{1} \frac{1}{\sqrt{1 + x^2}} \, dx = \left. \ln \left| \sqrt{1 + x^2} + x \right| \right|_{0}^{1} = \ln(2) \]

(c) Find the area of the region bounded by the curve \( y = e^{-x^2} \) and the x-axis from \( x = -\infty \) to \( x = \infty \).

\[ A = \int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi} \]

(d) Find the volume of the solid obtained by rotating the region bounded by the curve \( y = x^2 \) and the line \( y = 1 \) about the x-axis.

\[ V = \pi \int_{0}^{1} (1 - x^2)^2 \, dx = \frac{4\pi}{3} \]

(e) Find the mass of the solid obtained by rotating the region bounded by the curve \( y = \sin(x) \) and the line \( y = \cos(x) \) about the y-axis.

\[ M = \int_{0}^{\pi/2} \pi \sin^2(x) \, dx = \frac{\pi}{2} \]

(f) Find the moment of inertia of the solid obtained by rotating the region bounded by the curve \( y = x^3 \) and the line \( y = 2 \) about the x-axis.

\[ I = \int_{0}^{2} 2\pi y \sqrt{1 + (y')^2} \, dy = \int_{0}^{2} 2\pi y \left( \sqrt{1 + (3x^2)^2} \right) \, dy = \frac{16\pi}{5} \]

(g) Find the center of mass of the solid obtained by rotating the region bounded by the curve \( y = \sqrt{x} \) and the line \( y = 3 \) about the y-axis.

\[ \bar{x} = \frac{1}{M} \int_{0}^{3} x \pi \sqrt{x} \, dx = \frac{3}{2} \]

\[ \bar{y} = \frac{1}{M} \int_{0}^{3} \pi \left( \frac{y^2}{2} \right) \, dy = \frac{9}{4} \]

(h) Find the radius of gyration of the solid obtained by rotating the region bounded by the curve \( y = x^2 \) and the line \( y = 4 \) about the x-axis.

\[ k = \sqrt{I/\pi} = \sqrt{\frac{128\pi}{5}} \]

(i) Find the moment of inertia of the solid obtained by rotating the region bounded by the curve \( y = 1/x \) and the line \( y = 2 \) about the y-axis.

\[ I = \int_{1}^{2} 2\pi x \sqrt{1 + (y')^2} \, dx = \int_{1}^{2} 2\pi x \sqrt{1 + (\frac{1}{x^2})^2} \, dx = \frac{16\pi}{3} \]

(j) Find the moment of inertia of the solid obtained by rotating the region bounded by the curve \( y = x^3 \) and the line \( y = 1 \) about the y-axis.

\[ I = \int_{0}^{1} 2\pi x \left( \sqrt{1 + (y')^2} \right) \, dx = \int_{0}^{1} 2\pi x \left( \sqrt{1 + (3x^2)^2} \right) \, dx = \frac{16\pi}{5} \]

(k) Find the moment of inertia of the solid obtained by rotating the region bounded by the curve \( y = x^2 \) and the line \( y = 4 \) about the y-axis.

\[ I = \int_{0}^{1} 2\pi x \left( \sqrt{1 + (y')^2} \right) \, dx = \int_{0}^{1} 2\pi x \left( \sqrt{1 + (2x)^2} \right) \, dx = \frac{128\pi}{5} \]

(l) Find the moment of inertia of the solid obtained by rotating the region bounded by the curve \( y = \sin(x) \) and the line \( y = \cos(x) \) about the x-axis.

\[ I = \int_{0}^{\pi/2} 2\pi y \sqrt{1 + (y')^2} \, dy = \int_{0}^{\pi/2} 2\pi y \left( \sqrt{1 + (-\cos(x))^2} \right) \, dy = \frac{4\pi}{3} \]
2. (30 points) Determine if the following infinite series converge or diverge. In each case, state which test you use and show how the test works.

a. \[ \sum_{k=0}^{\infty} \frac{1}{k^{\pi}} \]

\( p \)-series with \( p = \pi > 1 \)
CONVERGENCE

b. \[ \sum_{k=0}^{\infty} \pi^k \]

Geometric Series
with \( r = \pi > 1 \) \( \Rightarrow \) DIVERGENCE

c. \[ \sum_{k=1}^{\infty} \frac{1}{k^{\pi}} \]

\( p \)-series with \( p = \pi > 1 \)
CONVERGENCE

d. \[ \sum_{k=0}^{\infty} k^{1/\pi} \]

\[ \sum_{k=0}^{\infty} \frac{1}{k^{1/\pi}} \]

\( p \)-series with \( p = \frac{1}{\pi} < 1 \)
DIVERGENT \( p \)-series

\[ \lim_{k \to \infty} k^{1/\pi} = \infty \neq 0 \]
DIVERGES
by \( n \)th Term Test

e. \[ \sum_{k=1}^{\infty} \left( \frac{1}{k} \right)^{\pi} \]

Geometric Series
with \( r = \frac{1}{\pi} < 1 \) \( \Rightarrow \) convergent

f. \[ \sum_{k=1}^{\infty} \frac{1}{k^{1/\pi}} \]

\( p \)-series with \( p = \frac{1}{\pi} < 1 \)
DIVERGENT \( p \)-series