Prof. Ron Buckmire

0. **The Final Exam in Math 120 is Friday May 9 6:30pm-9:30pm in Fowler 309**

1. **The ideas are the most important thing!**

2. **Practice some techniques.** Important techniques include determining the number of subdivisions needed to obtain a Riemann sum approximation of the definite integral of a monotone function to a given degree of accuracy; finding the derivative of an accumulation function; relating the graphs of a function, its derivative and its family of antiderivatives; writing the solution of an initial value problem as an accumulation function; using basic properties of integrals and antiderivatives; using the Fundamental Theorem of Calculus to evaluate definite integrals; approximating the value of a definite integral by approximating the integrand with a Taylor Series. In particular, you should know the table of antiderivatives/derivatives below.

a. **Techniques of Integration**

$f'(x) = F''(x)$	$f(x) = F'(x)$	$F(x) = \int f(x)dx$
0	1	$x + C$
nx^{n-1}	$x^n \ (n \neq -1)$	$\dfrac{1}{n+1}x^{n+1} + C$
$-\frac{1}{x^2}$	$\frac{1}{x}$	$\ln\lvert x\rvert + C$
$\cos(x)$	$\sin(x)$	$-\cos(x) + C$
$-\sin(x)$	$\cos(x)$	$\sin(x) + C$
$\sec^2(x)$	$\tan(x)$	$-\ln(\lvert\cos(x)\rvert) + C$
$2\sec^2(x)\tan(x)$	$\sec^2(x)$	$\tan(x) + C$
$-\dfrac{1}{(1+x^2)^2} \cdot 2x$	$\dfrac{1}{1+x^2}$	$\arctan(x) + C$
e^x	e^x	$e^x + C$
$a^x \ln(a)$	$a^x (a > 0)$	$\frac{1}{\ln(a)}a^x + C$
$\dfrac{1}{x}$	$\ln(x)$	$x\ln(x) - x + C$

b. finding antiderivatives and evaluating definite integrals by using u-substitution ($\int f'[g(x)]g'(x)\,dx = \int f'[u]\ du = f[u] + C = f[g(x)] + C$ pick $u = g(x)$ and convert the integral entirely to u variables. You can either return to x-variables or stay in u-space when evaluating definite integrals); However you should be able to convert one definite integral in x-variables entirely to another integral in u-variables given the particular u-substitution

c. finding antiderivatives and evaluating definite integrals by using integration by parts ($\int f \cdot g'\,dx = f \cdot g - \int f' \cdot g\,dx$ or $\int udv = uv - \int vdu$). Don't forget sometimes you have to do it more than once: Repeated integration by parts e.g. $\int x^2 e^{-x} dx$

4. Numerical Methods of integration:

a. Riemann Sums $\sum_{k=1}^{N} \mathtt{f(x_k)\Delta x}$,

b. Left Hand Sums $\mathtt{x_k} = \mathtt{a} + (\mathtt{k} - 1) * \Delta\mathtt{x}$

c. Right Hand Sums $\mathtt{x_k} = \mathtt{a} + \mathtt{k} * \Delta\mathtt{x}$, Midpoint method $\mathtt{x_k} = \mathtt{a} + (\mathtt{k} - .5) * \Delta\mathtt{x}$

d. Trapezoidal Rule $T = \frac{L+R}{2}$, Simpson's Rule $S = \frac{2}{3} M + \frac{1}{3} T$

e. Midpoint Error is always approximately -0.5 $\times$ Trapezoid Error (depends on concavity f'' and is proportional to N^{-2} or $(\Delta x)^2$)

f. Left Riemann Error versus Right Riemann Error (depends on slope f' and is proportional to N^{-1} or Δx)

g. Simpson Error (depends on $f^{(4)}$ and is proportional to N^{-4} or $(\Delta x)^4$)

Error Control e.g. Riemann error $= |f(b) - f(a)| \dfrac{b-a}{N} \leq$ ERROR $\Rightarrow$ solve for N

5. Applications of Integration

a. Finding the average value of a function on a given interval by setting up the proper integral over the proper interval, i.e. $\dfrac{1}{b-a} \displaystyle\int_a^b f(x)dx$

b. Finding area between curves (like $\displaystyle\int_a^b y_T(x) - y_B(x)\ dx$ (using vertical boxes) or $\displaystyle\int_c^d x_R(y) - x_L(y)\ dy$ (using horizontal boxes)

c. Finding solutions of initial value problems (differential equations with intial conditions) by separation of variables $\frac{dy}{dx} = f(x)g(y)$

d. finding the length of a curved segment of $f(x)$ from $(a, f(a))$ to $(b, f(b))$ by using the formula $L = \int_a^b \sqrt{1 + [f'(x)]^2} dx$

e. Finding volume by integrating cross-sectional area, i.e. $\int A(x)dx$

f. Finding volume of a solid of revolution around a particular axis

using washers $V = \displaystyle\int_a^b \pi R_{outer}^2 - \pi R_{inner}^2 dx$

(when rotated about the y-axis and R_{outer} and R_{inner} are functions of x) OR

$V = \displaystyle\int_a^b \pi R_{outer}^2 - \pi R_{inner}^2 dy$ (when rotated about the x-axis and R_{outer} and R_{inner} are functions of y)

using shells Volume=$V = \displaystyle\int_a^b (2\pi x) \cdot f(x) \cdot dx$

(when rotated about the y-axis)

OR $V = \displaystyle\int_a^b (2\pi y) \cdot g(y) dy$ (when rotated about the y-axis)

6. Other topics include:

   a. improper integrals of the first kind and of the second kind (remember the p-rules!)

   b. determining convergence of improper integrals using the Comparison Test for Improper Integrals

   c. polynomial approximations of a function near a point (Taylor polynomials), applications of Taylor polynomial approximations to derivatives and anti-derivatives, using calculus and algebra to find new Taylor Series from familiar ones

   d. tests for DIVERGENCE of an infinite series **(zero-limit); tests for convergence and/or divergence: alternating series, integral, comparison, absolute ratio, p-series and geometric series**

   e. useful series to remember are p-series, geometric series, harmonic series, alternating harmonic series. Remember that the sum of a geometric series $\sum_{k=0}^{\infty} ar^k$ converges to $\frac{a}{1-r}$ as long as the ratio $|r| < 1$

   f. Remember the differences and connections between improper integrals and infinite series and be able to articulate these concepts in written form

7. **Practice using tests for convergence.** Especially important are the Absolute Ratio Test and the Zero Limit Test for Divergence. Don't come into the exam without being able to take the limit as $k \to \infty$ of some expression involving k. You should be able to apply L'Hopital's rule on those indeterminate limits. Don't forget the other tests we have covered (the Integral Test, the Comparison Test, and Alternating Series Test).

8. Remember the basic idea of doing comparisons:
   If you want to show that something CONVERGES, you have to compare it to something which is **LESS THAN OR EQUAL TO** something you already know CONVERGES.
   If you want to show that something DIVERGES, you have to compare it to something which is **GREATER THAN OR EQUAL TO** something you already know DIVERGES.
   The "something" can either be an improper integral or an infinite series, but in either case the integrand or terms must all be POSITIVE. FUNCTIONS DO NOT converge or diverge, improper integrals or infinite series do.

8. Taylor Series To Remember...

$$
\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots &&= \sum_{k=0}^{\infty} \frac{x^k}{k!} \\
\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots &&= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \\
\cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots &&= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \\
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + \dots &&= \sum_{k=0}^{\infty} x^k \\
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} + \dots &&= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} \\
(1+x)^n &= 1 + nx + \frac{n(n-1)}{2!} x^2 + \dots &&= \sum_{k=0}^{\infty} \frac{n!}{k!(n-k)!} x^k
\end{aligned}
$$

7. **Evaluating Limits**

L'Hôpital's Rule

If $\lim_{x\to\infty} \dfrac{f(x)}{g(x)}$ is of the form $\frac{\infty}{\infty}$ or $\frac{0}{0}$ or $0 \cdot \infty$ then if if the limit $\lim_{x\to\infty} \dfrac{f'(x)}{g'(x)} = L$ exists, then $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = L$ In other words, if you have an indeterminate limit, just differentiate the numerator and denominator and take the limit again until you get a determinate answer. That answer will be the value of the limit.

There are more exotic indeterminate forms which look like 0^0, ∞^0, 1^∞ and 0^∞. In these you will have to use properties of logarithm: $\mathbf{B = EXP(\ LN(B)\ )}$ and $\ln(b^c) = c\ln(b)$ in order to arrange the expression into a fraction that you can apply L'Hopital's Rule to.

$\lim_{x\to\infty} f(x)^{g(x)} = \lim_{x\to\infty} e^{g(x)\ln(f(x))} = e^{\lim_{x\to\infty} g(x)\ln(f(x))}$.

You should be comfortable with discounting or ignoring parts of an expression when those parts are getting very small compared to the rest of the expression.

Remember, $\sin(x)$ and $\cos(x)$ only return values between ± 1.

Limit Formulas To Remember

$$\int_a^\infty \frac{dx}{x^p} = \begin{cases} \textbf{DIVERGES} & \text{when } p \le 1 \\ \textbf{CONVERGES} & \text{when } p > 1 \end{cases}$$

$$\int_0^b \frac{dx}{x^q} = \begin{cases} \textbf{DIVERGES} & \text{when } q \ge 1 \\ \textbf{CONVERGES} & \text{when } q < 1 \end{cases}$$

$$\lim_{x\to\infty} e^{kx} = \begin{cases} 0 & \text{when } k < 0 \\ \infty & \text{when } k > 0 \end{cases}$$

$$\lim_{x\to\infty} x^r = \begin{cases} 0 & \text{when } r < 0 \\ \infty & \text{when } r > 0 \end{cases}$$

$$\lim_{x\to 0^+} x^r = \begin{cases} \infty & \text{when } r < 0 \\ 0 & \text{when } r > 0 \end{cases}$$

Remember that when there's a race between e^x and any polynomial function x^p as $x \to \infty$, e^x will always win. Conversely, $\ln(x)$ will lose any race with x^p as $x \to \infty$

8. **Interval Of Convergence and Radius Of Convergence**

   Consider $\sum_{k=0}^{\infty} b_k(x-a)^k$. This Power Series may not converge for all x-values. The set of x-values for which the series converges is called the *interval of convergence*. The interval of convergence is always centered on the point a.

   The interval of convergence can be infinite, i.e. $(-\infty, \infty)$ a.k.a. "all Real Numbers" Or it can be a finite interval of the form $(a-R, a+R), [a-R, a+R], (a-R, a+R]$ or $[a-R, a+R)$. R is called the *radius of convergence*. We use this idea of interval of convergence in the SPECIFIC EXAMPLE of trying to determine for which x values a Taylor Series will converge. (Remember a Taylor Series is just a special case of Power Series.) You can use the formula $\frac{1}{R} = \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|$ on the coefficients when the Taylor Series looks like $\sum_{k=0}^{\infty} c_k x^k$ or you can always just use the Absolute Ratio Test on the entire series.

9. **Fourier Series** Fourier Series are polynomials of **sine** and **cosine** of infinite degree. A Fourier series is used to approximate a *periodic* function on an infinite interval. The coeefficients of the series are determined by INTEGRATION (how is this different from Taylor Series?).

   In general, a Fourier Series is used to approximate a function $f(t)$ with period P

$$f(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(\frac{2k\pi}{P}t) + \sum_{k=1}^{\infty} b_k \sin(\frac{2k\pi}{P}t)$$

   where

$$\begin{aligned} a_0 &= \frac{1}{P}\int_0^P f(t)\ dt \\ a_k &= \frac{2}{P}\int_0^P f(t)\ \cos(\frac{2k\pi}{P}t)\ dt \\ b_k &= \frac{2}{P}\int_0^P f(t)\ \sin(\frac{2k\pi}{P}t)\ dt \end{aligned}$$

   Usually, the function is periodic on 2π so that the Fourier Series simply looks like

$$f(t) = a_0 + a_1\cos(x) + b_1\sin(x) + a_2\cos(2x) + b_2\sin(2x) + +a_3\cos(3x) + b_3\sin(3x)...$$

   Recall that COSINE is an EVEN function (i.e. $\cos(x) = \cos(-x)$ for ANY x) and that SINE is an ODD function ($-\sin(x) = \sin(-x)$ for ANY x).

   Therefor if the function you are approximating is an ODD function it's Fourier Series will only contain SINE functions. If the function being approximated is an EVEN function its Fourier Series will only contain COSINES (and the CONSTANT a_0).

   a_0 the first term of the Fourier Series is always exactly equal to the average of the given function on its entire period.

FINAL EXAM BLUE NOTES

Name: ______________________________

SAMPLE PROBLEMS FOR MATH 120 SPRING 2014 FINAL EXAM

1. Draw the area represented by the following integral

$$\int_0^4 \frac{1}{1+x^2}\, dx.$$

Using $n = 4$ subintervals, estimate the definite integral using the following:

a. left endpoint Riemann sum
b. right endpoint Riemann sum
c. midpoint Riemann sum
d. trapezoid rule and
e. Simpson's rule.
f. Compute the exact integral (use the FTC).

Then compare your estimates. Know which ones are most accurate and why.

2. a. Using integration by substitution, find $\int x\sqrt{1+3x}\, dx$

b. Using integration by parts, find $\int \frac{\ln(x)}{x^2}\, dx$

c. Using integration by parts or integration by substitution, find $\int \sin^2(x)\cos x\, dx$

d. By evaluating a definite integral, find the area under the x-axis but above the curve $y = x^2 - 3x$. Draw a sketch of the curve and indicate the requested area on your sketch.

e. Below is a list of indefinite integrals. Find an antiderivative for each.

$$\int \sin(2x)dx, \quad \int 3\cos^2(4x)\sin(4x)\; dx, \quad \int 7^x\; dx, \int x^2(x^3-6)^{20}\; dx, \quad \int \frac{3}{4x-2}\; dx.$$

f. Find the average value of $f(x) = \sin^2(3x)\cos(3x)$ on $[0, \frac{\pi}{6}]$.

g. Evaluate the following:

$$\int_0^1 \frac{x}{\sqrt{1+x}}\,dx, \quad \int \frac{x}{1+\sqrt{x}}\,dx.$$

3. Explain why using Simpson's method to evaluate the definite integral in part (d) above will compute the answer **exactly**, but if you were to use the Midpoint or Trapezoid Method the answer would only be approximate. (You can test this for yourself by trying to evaluate the definite integral using Midpoint, Trapezoid and Simpson's Method and seeing that Simpson's is exact.)

4. Calculate the following (improper) integrals.

a. $\displaystyle\int_0^1 \ln(2x)\,dx$

b. $\displaystyle\int_0^6 \frac{1}{\sqrt{6-x}}\,dx$

c. $\displaystyle\int_3^8 \frac{1}{2-x}\,dx$

d. $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}+2}\,dx$

e. $\displaystyle\int_1^\infty \frac{x^2}{1+\sin^2(x)}dx$

f. $\displaystyle\int_1^\infty \frac{\sin^2(x)}{1+x^2}dx$

5. Calculate the area of the region under the curve $y = \sqrt{x}+1$, above the x-axis and between $x = 0$ and $x = 4$.

6. Find the unique solution to the IVP

$$y' + y(1+x) = 0, \qquad y(-2) = 1$$

7. Given that $A(\mathcal{X}) = \int_1^{\mathcal{X}} \sqrt{1+e^{2x}}\,dx$

   a. Evaluate $A(1)$
   b. Evaluate $A'(1)$
   c. Evaluate $A''(1)$
   d. Show that $A(b)$ represents the length of the curve $y = e^x$ from the coordinate $(1, e)$ to (b, e^b)
   e. If $B(\mathcal{X}) = \int_1^{\sin(\mathcal{X})} \sqrt{1+e^{2x}}\,dx$, Find $B'(\mathcal{X})$.

8. Compute the following:
   $A = \int_0^2 x^2 t\,dx$
   $B = \int_0^2 x^2 t\,dt$
   $C = \int_0^2 x^2 t\,dk$
   $D = \int_0^x k^2 t\,dt$
   $E = \int_0^k x^2 t\,dx$
   What is $\frac{dB}{dx}$ equal to? What about $\frac{dD}{dx}$? $\frac{dE}{dx}$? $\frac{dA}{dt}$?

9. Solve the following initial value problem

$$f'(x) = \frac{3}{x} + e^x + x^3, \qquad f(2) = 0.$$

10. a. Find the second degree Taylor polynomial based at $a = 2$ for the function $g(x) = e^{2-x}$.

    b. Use your answer in part (a) to estimate

$$\int_{1.9}^{2.1} e^{2-x}\,dx.$$

11. Determine whether the following series converge or diverge.

a. $\sum_{n=0}^{\infty} \frac{1}{3+4^n}$

b. $\sum_{n=0}^{\infty} \frac{6^n}{n!}$

c. $\sum_{n=1}^{\infty} (-1)^n \frac{5}{\sqrt{n^3}}$

d. $\sum_{n=1}^{\infty} (-1) \frac{2}{\sqrt{n}}$

12. Evaluate the following sums exactly:

$$1 + \frac{1}{e} + \frac{1}{e^2} + \frac{1}{e^3} + \cdots.$$

$$1 - \frac{1}{e} + \frac{1}{e^2} - \frac{1}{e^3} + - \cdots.$$

$$1 + 2\frac{1}{e} + 3\frac{1}{e^2} + 4\frac{1}{e^3} + \cdots.$$

$$1 + \frac{1}{e^2} + \frac{1}{e^4} + \frac{1}{e^6} + \cdots.$$

14. Write down the Taylor Series for $f(x) = e^{-x^3}$ about the point $x = 0$. (HINT: You probably do **not** want to do this by taking any derivatives of $f(x)$.)

15. Find the radius and interval of convergence of the infinite series $\sum_{k=0}^{\infty} (-4)^k x^{2k+1}$

16. Find the Fourier Series representation of the following function $f(x)$ with period 2π defined below:

$$f(x) = \begin{cases} 0, & \text{when } -\pi < x \leq -\pi/2 \\ 1, & \text{when } -\pi/2 < x \leq \pi/2 \\ 0, & \text{when } \pi/2 < x \leq \pi \end{cases}$$

17. **Stewart, page 422, #17, #19, #20**. Describe the solid whose volume is given by the following integrals:
$\int_0^{\pi/2} 2\pi x \cos(x) dx$, $\int_0^{\pi} \pi(2 - \sin x)^2 dx$ and $\int_0^4 2\pi(6-y)(4y-y^2) dy$

18. **Stewart, page 422, #14**. Let $\mathcal{R}$ be the region in the first quadrant bounded by the curves $y = x^3$ and $y = 2x - x^2$. Show that
(a) The area of $\mathcal{R}$ is $\mathbf{5/12}$.
(b) The volume obtained by rotating $\mathcal{R}$ about the x-axis is $\mathbf{41\pi/105}$
(c) The volume obtained by rotating $\mathcal{R}$ about the y-axis is $\mathbf{13\pi/30}$