Summation Notation, Area, Riemann Sums
Class 3: **Monday January 27**

Sigma Notation

The formula $\frac{12}{N}5^2\frac{(1^2+2^2+3^2+\cdots+N^2)}{N^2}$ can be written more compactly as $\frac{12}{N^3}5^2\sum_{k=1}^{N}k^2$.

$\sum_{k=1}^{6}k = 1+2+3+4+5+6$ and in general $\sum_{k=1}^{N}k = 1+2+3+\cdots+N-1+N = \frac{N}{2}(N+1)$

Can you come up with a similar formula for $\sum_{k=1}^{N}k^2 = 1^2+2^2+3^2+\cdots+N^2$ in terms of N?

<u>Examples</u>

1. $\sum_{k=3}^{11}\frac{1}{k} =$

2. $\sum_{j=-3}^{3} j+6 =$

3. $\sum_{k=0}^{4}\sin(\frac{k\pi}{2}) =$

4. $\sum_{i=0}^{\infty}\frac{1}{2^i} =$

Quadrature Again: Area under a curve

Suppose we want to find the area of the region R described below:
$0 \leq x \leq 2$, above the x-axis, and below $f(x) = x^2 + 1$.

Below, sketch and indicate the region R described above.

5. What is the area of the largest rectangle which will fit completely underneath the curve $f(x) = x^2 + 1$?

6. In the shaded region draw two rectangles of equal width, neither of which goes above the curve $f(x) = x^2 + 1$.

The width of each rectangle is _____.
The heights of the two rectangles are _____ and _____.
The sum of the areas of the two rectangles is _____.

7. Find a better estimate for the area of the shaded region by repeating the above procedure, but this time with FOUR rectangles.

8. Repeat the estimate with 100 rectangles! Just write the appropriate expression, but do not evaluate it. Try to use Sigma Notation. This expression is an example of a **Riemann Sum**.

9. Write down the estimate using N rectangles, instead of 100.

10. The larger N is, the more __________ this estimate is. So, the EXACT value of the area of region R is:

A shorthand way of writing this is: $\int_0^2 x^2 + 1dx$

This is called the **definite integral** of the function $f(x) = x^2 + 1$ from $x = 0$ to $x = 2$.

WARNING: The definite integral gives the *negative* of the area if the function is below the x-axis.

Example: Find the value of:

$$\int_0^3 (-2x)\, dx =$$

Riemann Sums

In general a Riemann Sum must have the format

$$\sum_{k=1}^{N} f(x_k)\Delta_k = f(x_1)\Delta_1 + f(x_2)\Delta_2 + \cdots + f(x_N)\Delta_N$$

Above, we were always doing **left-hand** Riemann Sums. This means:

In each "sub-interval" we used the _______ endpoint to calculate the height of each rectangle.

There was no special reason to use LEFT-hand sums. We could just as well use RIGHT-hand Riemann Sums, and everything would still work the same (except in this example we'd happen to get over-estimates).

In the LIMIT, the right-hand sums would still give the same exact number as the left-hand sums do.

Homework Assignment #3
(*3 points* due Friday January 31)

1. Find the area bounded by the curve $f(x) = x^2 + 1$, $x = 0$, $x = 2$ and $y = 0$ using Right-hand Riemann Sums with $N = 1$, $N = 2$, $N = 4$ and $N = 10$. (You will probably need to use your calculator of a PC with Microsoft Excel.)

2. Increase the valaue N until you are confident that you know what the exact area of the region R is to TWO decimal places. Write down that value.

3. (BONUS: 1 point) Can you propose a rule to determine when using Right-hand Riemann Sums will produce and over-estimate to the exact area, and when they will produce and under-estimate to the exact area?