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Lab #8
Math 120 Lab
Thursday
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Visualizing Taylor Series and Fourier Series Approximations

§1 Taylor Series

For many functions $f(x)$, there is a standard scheme for developing a power series which converges to the function. This series is called the *Taylor series* for the function. You have seen in class that the Taylor series for the function $f(x)$ centered at $a=0$ is given by

$$\sum_{k=0}^{\infty} c_k x^k, \quad \text{where } c_k = \frac{f^{(k)}(0)}{k!}.$$

More generally, the Taylor series for the function $f(x)$ centered at $x=a$ is given by

$$\sum_{k=0}^{\infty} c_k (x-a)^k, \quad \text{where } c_k = \frac{f^{(k)}(a)}{k!}.$$

The examples in this lab exercise will all be Taylor series centered at $x=0$. Because the calculation of the coefficients of the Taylor series is simple in principle (but messy in practice), it is an ideal situation to use a computer. The computer application `derive` has a built-in utility to generate the terms in the Taylor series.

Geometric series

The Taylor series for the function $f(x) = \dfrac{1}{1-x}$ has a particularly simple form.

1. Start up `derive` and `Author` the expression for the function $f(x)$ defined above. Plot the graph of this function and `Tile` (in the `Window` menu) the algebra and plot windows so they are side by side.

2. Use hand calculation to determine the slope of the graph $y = \dfrac{1}{1-x}$ at $a=0$. Then use the `Calculus|Taylor` box in `derive` to generate the formula for the degree 1 Taylor polynomial centered at $a=0$. (Note on terminology: `derive` calls this the "order 1 Taylor series with expansion point 0.") Then plot the line and check that your hand-generated slope agrees.

3. Using the same function and expansion point, generate the Taylor polynomials for $f(x) = \dfrac{1}{1-x}$ of orders $2, 3, 4,$ and 10. Plot each one in the same window. What do you see?

4. As you have seen, the Taylor polynomial of degree N is often called $P_N(x)$, when the center point (here $a=0$) is established by the context. What do the graphs suggest about the values of the following limits?

$$\lim_{N\to\infty} P_N(0.4), \qquad \lim_{N\to\infty} P_N(-0.4)$$

5. What do the graphs suggest about the following limits?

$$\lim_{N\to\infty} P_N(1), \qquad \lim_{N\to\infty} P_N(-1)$$

6. Write out (with or without using the summation symbol) the Taylor *series* for the function $f(x) = \dfrac{1}{1-x}$ centered at $a=0$.

7. Based on the graphs you have generated, is it *always* true that the Taylor series for $f(x) = \frac{1}{1-x}$ converges to the function $f(x)$? Why or why not?

8. For the function $f(x) = \dfrac{1}{1-x}$, for which x does the Taylor series centered at $a=0$ converge to $f(x)$? What is the connection with geometric series? (Hint: Is the Taylor series a geometric series?)

§2 Fourier Series

Suppose that $f(x)$ is a periodic function with period 2π. If we want to approximate this function with a trigonometric polynomial of degree n,

$$F_n(x) = a_0 + a_1\cos(x) + b_1\sin(x) + a_2\cos(2x) + b_2\sin(2x) + \cdots + a_n\cos(nx) + b_n\sin(nx)$$

then the "best" coefficients to use are the following **Fourier coefficients**:

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ dx \\ a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\ dx \\ b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\ dx \end{aligned}$$

where $k \geq 1$.

Whereas a Taylor Series attempts to approximate a function locally about the point where the expansion is taken, a Fourier series attempts to approximate a periodic function over its entire domain. That is, a Taylor series approximates a function pointwise and a Fourier series approximates a function globally.

Error in Using Taylor Series and Fourier Series
The error in approximating a function $f(x)$ near a point a by a Taylor Polynomial $P_{n,a}(x)$ is given by the absolute value between the difference between the function and the Taylor Polynomial at a point x:

$$e_n(x) = |P_{n,a}(x) - f(x)| = C\frac{(x-a)^{n+1}}{(n+1)!}$$

where C is a constant related to $f^{(n+1)}(x)$, the $n+1$-st derivative of $f(x)$.

Previously we showed graphically that the interval on the x-axis for which $e_n(x)$ is very close to zero gets bigger and bigger as the degree of the Taylor polynomial, n increases.

The error in approximating a 2π-periodic function $f(x)$ by a Fourier polynomial $F_n(x)$ is given by the following integral:

$$E_n = \int_{-\pi}^{\pi} |f(x) - F_n(x)|^2\ dx.$$

That is, $F_n(x)$ is a "good" approximation of $f(x)$ if E_n is small. Notice that E_n does not depend on x. E_n gives a measure of how well $F_n(x)$ approximates $f(x)$ over all of $[-\pi, \pi]$, not just at a single point as our error bound for Taylor polynomials did. The above Fourier coefficients are the "best" coefficients in the sense that those coefficients make E_n as small as possible.

Part 1: Computing Fourier series

Find the first three Fourier polynomials, $F_1(x), F_2(x)$, and $F_3(x)$ for the square wave function of period 2π given by:

$$g(x) = \begin{cases} -1 & -\pi \le x < 0 \\ 1 & 0 \le x < \pi \end{cases}$$

(Hint: a_0 and the rest of the a_k's are easy to determine with a bit of thought. Focus on the graph of the function and think about the area. $F_2(x)$ is also easy to find with no computation.)

Now write down a general formula for the Fourier coefficients a_k and b_k. You may want to break the integral up into two parts to compute b_k. You can use Derive's integration capabilities to check your answer.

Enter a general formula for the nth Fourier polynomial of $g(x)$ in Derive as follows:

Author $b(k)$:= the formula you found above

Author $F(n, x) := \text{SUM}(b(k) * \sin(kx), k, 1, n)$

Now Author, Simplify, and Plot each of the following Fourier polynomials. In each case write down the Fourier series and it's draw it's graph on the interval $[-\pi, \pi]$.

Fourier Polynomial	Sketch of Approximation
F(1,x)	
F(3,x)	
F(5,x)	
F(17,x)	

Enter $g(x)$ into Derive as follows: **Author** $g(x)$:= -1 chi(-pi,x,0) + 1 chi(0,x,pi)

Now Plot $g(x)$ in the same axes as $F_n(x)$ for $n = 1, 3, 5,$ and 17 in order to get a rough idea of the accuracy of the Fourier approximations.

Lab Report

There is no written lab report which needs to be handed in for this lab but you should understand the main ideas and skills in this lab, and be able to answer the following questions.

Question 1 True or False? The Taylor series for every function $f(x)$ converges to $f(x)$ for every value of x.

Question 2 Show in detail how you computed the coefficients for the Fourier series of $g(x)$. Note, the function $g(x)$ is odd. Define an odd function and explain how $g(x)$ being odd simplifies the computation of the a_k coefficients. Give the definition of an even function and explain how the computation of the b_k Fourier coefficients is simplified for an even function. **Remembering how the computation of the Fourier coefficients simplifies when the function is odd or even will save you a lot of work/time in the future.**

BONUS QUESTION (Extra Credit Homework #4)

Determine for which values of x, the series $\displaystyle\sum_{k=1}^{\infty} \frac{\sin(kx)}{k}$ converges. Do you see any connections between this series and today's lab?

Solutions to this question are DUE Thursday May 1 for 5 bonus points