Names: ____________________

____

____

Lab #6
Math 120 Lab
Thursday
April 3, 2003
Ron Buckmire

Convergence Tests for Infinite Series

Today we are going to look at a few different tests for convergence of an infinite series of numbers. We will try to familiarize ourselves with methods which determine whether a particular infinite series converges.

SECTION A: Warm-Up

In the computer program *Derive* we can express a sum in two different ways.

I. One way that we can define a sum, whether infinite or finite, is to use the **C**alculus and **S**um options. Having chosen these options, the computer will ask you for a few pieces of information: (1) the expression to be summed (type in "k^2") – remember this represents a list of numbers, there is no "function variable" here; (2) the variable of summation – this is **NOT** a variable inside the sum, it is the variable which is "counting" through the integers for you (type in "k"); (3) the lower limit (type in "1"); (4) and the upper limit (type in "3"). When you hit OK you should see this sum written out for you. Write out this sum and calculate it **before** you ask the computer to do it.

$$\sum_{k=1}^{3} k^2 =$$

Tell the computer to approximate it or simplify it. Do you get the same result? You should.

II. A second way that we can define a sum is by **A**uthoring the sum ourselves. **A**uthor the following:

SUM (k, k, 1, 3)

SUM (1/k^2, k, 1, 3)

Do you see the pattern? You can also **A**uthor a function involving a sum. Make sure you use := not just = (it means you are "defining", or "assigning to" the function $S(n)$).

S(n) := SUM(k, k, 1, n)

Do you see what it does? What is S(3)? After you figure it out, **A**uthor S(3) and ask the computer to **S**implify it.

This last way of defining the sum as a function of n may seem tedious, but if you have to sum up the same thing over and over and over again, just changing the upper limit each time, then this is a great way to do it!

You can also ask Derive to solve the problem directly by putting in the sum an upper limit of ∞ by typing in "inf". Do this and see what the computer gives you, both symbolically and as an approximation. (Un)fortunately the computer will not always give you a meaningful answer if you put the upper limit as infinity, so you may have to use other methods to determine convergence of the series you are interested in.

SECTION B: Convergence Tests

I. The "Zero-Limit" Test for Divergence

If $\lim_{n\to\infty} a_n \neq 0$ (or does not exist) then the series $\displaystyle\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + a_4 + \ldots$ **diverges**.

We can rephrase this as: **IT IS TRUE** that if $\displaystyle\sum_{k=1}^{\infty} a_k$ CONVERGES, then $\lim_{n\to\infty} a_n = 0$.
Do you see why this "rephrasing" is really saying the exact same thing as its previous sentence?

But **IT IS NOT TRUE** that if $\lim_{n\to\infty} a_n = 0$ then $\displaystyle\sum_{k=1}^{\infty} a_k$ necessarily CONVERGES.

II. Integral Test for Convergence and Divergence

This test relates facts about improper integrals to facts about infinite series.

Suppose $f(x)$ is a continuous and decreasing function and $f(x) > 0$ for all $x \geq 1$. Let $a_k = f(k)$. THEN
(a) If the $\displaystyle\int_1^{\infty} f(x)dx$ CONVERGES, then the infinite series $\displaystyle\sum_{k=1}^{\infty} a_k$ CONVERGES.
(b) If the $\displaystyle\int_1^{\infty} f(x)dx$ DIVERGES, then the infinite series $\displaystyle\sum_{k=1}^{\infty} a_k$ DIVERGES.

Definition For $p > 0$ the infinite series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ is called a p-series.

If one applies the integral test to the p-series then
if $p \leq 1$, then p-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ DIVERGES.
If $p > 1$, then the p-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ CONVERGES.

III. Comparison Test for Convergence and Divergence

(a) If $0 \leq b_k \leq a_k$ for each k and $\sum_{k=1}^{\infty} a_k$ converges, then $\sum_{k=1}^{\infty} b_k$ also CONVERGES.

(b) If $0 \leq a_k \leq c_k$ for each k and $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} c_k$ also DIVERGES.

IV. Absolute Ratio Test

For any infinite series $\sum_{k=1}^{\infty} a_k$, if

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$$

then $\sum_{k=1}^{\infty} a_k$ CONVERGES.

If $L > 1$ or if $|a_{k+1}/a_k|$ does not exist, then $\sum_{k=1}^{\infty} a_k$ DIVERGES

If $L = 1$ the test is INCONCLUSIVE.

V. Alternating Series Test (Skip for now, we will go into this next week)

Definition An infinite series is said to be an **alternating series** if it has the form $\sum_{k=1}^{\infty} (-1)^k a_k$ or $\sum_{k=1}^{\infty} (-1)^k a_k$ where $a_1, a_2, a_3, \ldots$ are all positive numbers.

If $a_1, a_2, a_3, \ldots, a_k, \ldots$ is a sequence of **decreasing positive numbers** such that $\lim_{n \to \infty} a_n = 0$ then the alternating series $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ CONVERGES.

SECTION C: Examples Galore

Now that we have listed a number of tests for convergence, the point of this lab is to have you consider the following infinite series and try and determine whether they converge or not.

Directions: For each of the following series, do the following:

- Write out the first few (about 3) **terms** of the series
- determine what kind of series it is (alternating, increasing, decreasing, positive, etc)
- Apply Test **I.** for Divergence on the series (Zero-Limit test)
- See if `Derive` can help you determine the convergence directly
- Determine convergence by using one of the other tests

1. $\displaystyle\sum_{k=1}^{\infty} 4$

2. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)}{k}$

3. $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{k^2}$

4. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$

5. $\sum_{k=1}^{\infty} \frac{k}{\sqrt{k+5}}$

6. $\sum_{k=1}^{\infty} k^{1.4141}$

7. $\sum_{k=1}^{\infty} \frac{1}{k^2+1}$

8. $\sum_{k=1}^{\infty} \frac{2}{k(k+1)}$

9. $\displaystyle\sum_{k=1}^{\infty} \frac{\ln(k)}{k^3}$

10. $\displaystyle\sum_{k=1}^{\infty} ke^{-k}$

11. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2^k}$

12. $\displaystyle\sum_{k=1}^{\infty} k\cos(k\pi)$

BONUS Write-up: Write up very neat solutions to the above twelve (12) problems. You should discuss your results. You should try to make connections between the various series if you can. You should try to think about what makes an infinite series converge or not converge.

Due: One week from today, in lab: April 10 .