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Lab #1
Basic Calculus 2
January 30, 2003

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Understanding and Computing Riemann Sums

Let us begin by reviewing what we already know about Riemann sums. A particular Riemann sum is the result of many choices. (1) There is the function and the interval:

$$f(x), \qquad x \in [a, b];$$

(2) there is the number of subintervals of the interval and the sizes of each of the subintervals, also called the "**partition**" of the interval:

$$N, \qquad \Delta x_1, \Delta x_2, \ldots, \Delta x_k, \ldots, \Delta x_N;$$

and (3) there is the sequence of "sampling" points, one chosen from each of the subintervals:

$$x_1, x_2, \ldots, x_k, \ldots, x_N.$$

The **Riemann sum** for the function f on the interval $[a, b]$ with this partition and this set of sampling points then is:

$$\begin{aligned} \text{SUM} &= f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + \ldots + f(x_k)\Delta x_k + \ldots + f(x_N)\Delta x_N \\ &= \sum_{k=1}^{N} f(x_k)\Delta x_k \end{aligned}$$

When we wish to approximate some quantity using Riemann sums, we usually create a partition, pick our sampling points (usually with some scheme in mind – always the highest point, the lowest point, the midpoint, the right-hand endpoint, etc.), and make an approximation. To get a better approximation, we simply refine our partition by taking more subintervals.

A. Circle of radius 1

You know that the circle centered at the origin in the xy-plane with radius r has the equation $x^2 + y^2 = r^2$. Denote the area of this circle as $A(r)$.

1. Give the equation $f(x)$ for the function whose graph is the semicircle of radius 1 centered at the origin which lies in the first and second quadrants. Sketch the diagram.

2. Denote by Q the region inside the above circle which is in the first quadrant only. Explain why the following expression is *greater than* the area of Q.

$$f(0)(0.2) + f(0.2)(0.2) + f(0.4)(0.2) + f(0.6)(0.2) + f(0.8)(0.2)$$

3. Write a sum which gives the corresponding *underestimate* of the area of Q.

4. What is the "spread" (difference) between the two estimates above?

5. Calculate, as decimals, the two estimates above. Organize your work into a table:

6. Write the expression which overestimates the area of Q using ten rectangles of width 0.1.

7. For the generic positive integer N, give the expression which gives an overestimate of the area of Q using N rectangles of equal width.

8. Open the Excel spreadsheet SUMS1. It shows the calculation of the overestimate using 5 rectangles; check that it agrees with your work above. Then modify the spreadsheet to calculate the overestimate using 10 rectangles.

9. Take a look at how the spreadsheet uses the "`@SUM()`" function to calculate the overestimate. Modify the expression to calculate the underestimate. Write the underestimate as a summation here, and use the spreadsheet to caculate its value.

   Use the work above to give another estimate for the value of $A(1)$. (Record both the upper and lower estimates.)

10. Now vary the method using 20 rectangles, then 100 rectangles. Record all your estimates here.

11. Use the spreadsheet to give both overestimates and underestimates that are guaranteed to be within 0.005 of the true value of the area of Q. Use this to estimate the value of $A(1)$ which, of course, is usually denoted π.

    How large does N have to be for you to obtain the value of $A(1)$ with an error less than 0.005?

Volume of a Square Pyramid

Look at the third tab of the spreadsheet marked "Volume." This sheet will help you to produce increasingly accurate approximations of the volume of a square pyramid with height H and square base b. We know that the expression for using N boxes to overestimate the volume is given by $\frac{b^2 H}{N^3} \sum_{k=1}^{N} k^2$. The underestimate for the volume of the pyramid involves subtracting the volume of the largest box at the base of the pyramid from the overestimate, producing the expression $\frac{b^2 H}{N^3} \sum_{k=1}^{N-1} k^2$

We will use the Excel spreadsheet to compute $\sum_{k=1}^{N} k^2$.

1. First produce a column A which contains the integers $k = 1, 2, 3, \cdots, N$.
2. Secondly, produce a column B which contains the integers $k^2 = 1, 4, 9, 16, 25, \cdots, N^2$ (You should be able to come up with a relationship between this column and the previous column).
3. Thirdly, add up all the entries in column B to obtain a value for $\sum_{k=1}^{N} k^2$
4. Compare your computed answer of $\sum_{k=1}^{N} k^2$ to the expression $\frac{N(N+1)(2N+1)}{6}$. Are they the same or different?
5. Use the above results to fill in the number for **Squared Integer Sum** in the Volume spreadsheet and then compute the overestimate and underestimate for the volume of a pyramid with $H = 12$ and $b = 5$ using $N = 1, 2, 4, 6, 10, 100, 200, 500$. What is the *exact* value of the volume of this pyramid?
6. Let $H = 1$ and $b = 1$. Increase the number of N until you obtain a value for the volume of this pyramid.
7. Can you use the above results to produce a formula for the volume of a pyramid which uses the height H and base b? Confirm your formula by using the spreadsheet and your formula to compute the exact volume of the pyramid with $b = 7$ and $H = 9$.

Writing up this Lab.

Your lab group should turn in one neat copy of this lab worksheet with accompanying Excel printouts to illustrate your computations. All members of the group must sign the lab coversheet. This lab is due **Thursday February 6**.