The first part of this lab uses *Derive* to explore the relationships between the derivatives of a function and the graph of the function. Graphical relationships between a function and its inverse are also examined.

A Function and Its Inverse

Consider the function $f(x) = \sin(x)$, $-\pi/2 \leq x \leq \pi/2$.

To begin, **Declare** the Function f(x) with formula **Definition sin(x)**. Then plot this function. Sketch the plot below and explain why this function has an inverse. (Note: Just plot $\sin(x)$ and sketch the graph over the indicated domain.)

The inverse of this function is *denoted* two different ways. The first, using the standard notation for inverse functions, is $f^{-1}(x) = \sin^{-1}(x)$. The second uses an older terminology taken from trigonometry: $f^{-1}(x) = \arcsin(x)$. These are just two different names for the same function. *Derive* uses a notation consistent with the older terminology.

Now **Declare** the Function g(x) with formula **Definition asin(x)**. Confirm that $f(g(x)) = x$ by **Author**ing and **Simplify**ing **f(g(x))**. Plot $g(x)$ on top of the graph of $f(x)$, and add this to your sketch above.

Note that there is one value of x for which $g(x) = f(x)$. What this value? You may either use your knowledge of trigonometry or the *Zoom* feature of *Derive* to help answer this question.

Now zoom in on the point where the graphs intersect until you can estimate the slopes of each graph at this point. What are the values of these slopes? (Realizing that these are the graphs of f and f^{-1}, you may be able to determine these slopes exactly.)

Evaluate $b = \sqrt{3}/2$, then use **Option: Trace Mode** to move the cursor to the point $(b, a) = (\sqrt{3}/2, g(\sqrt{3}/2)$ on the graph of g. **Center on the cross**, then **Zoom in** until you can obtain a good estimate of $g'(b)$.

Repeat, this time moving the cursor to $(a, b) = (g(\sqrt{3}/2), \sqrt{3}/2)$ on the graph of f. Zoom in to estimate $f'(a)$.

Compare your estimates of $g'(b)$ and $1/f'(a) = 1/f'(g(b))$. Does this agree with the theory? What could you do to improve these results?

You know that if $f(a) = b$ and $g(x) = f^{-1}(x)$, then

$$g'(b) = 1/f'(a) = 1/f'(g(b)), \quad \text{provided } f'(a) \neq 0.$$

Using this result and the trigonometric identity $\cos^2(x) + \sin^2(x) = 1$ to obtain a formula for the derivative of $\arcsin(x)$.

Check: $\dfrac{d}{dx}\arcsin(x) = \dfrac{1}{\sqrt{1-x^2}}$.

Confirm this using **Calculus Differentiate** in *Derive*.

Now examine your graphs of $f(x)$ and $g(x) = f^{-1}(x)$ once more. Complete the following observations:

As $x \to \dfrac{\pi}{2}^{-}$, $f'(x) \to$

As $x \to -\dfrac{\pi}{2}^{+}$, $f'(x) \to$

As $x \to 1^{-}$, $g'(x) \to$

As $x \to -1^{+}$, $g'(x) \to$

Based on these observations, formulate a general statement about points where the derivative of an inverse function becomes infinite. How can your assertion be proven?

Inverse Trigonometric Functions and Trigonometric Substitutions

Consider the right triangle below with hypoteneuse a and "opposite" side length x. In terms of a and x, what is the length of the "adjacent" side?

In terms of a and θ, what is x?

Use these two results to determine the length of the adjacent side in terms of a and θ.

In terms of a, θ and $d\theta$, what is dx?

Use your work thus far to rewrite the antiderivative below in terms of a, θ and $d\theta$ rather than in terms of a, x and dx. Then obtain a formula for the antiderivative in terms of a and θ:

$$\int \frac{dx}{\sqrt{a^2 - x^2}} =$$

Use an appropriate inverse trigonometric function to express θ in terms of x and a.

Use this result to express the antiderivative above in terms of x and a.

Check that this antiderivative is correct by differentiating! Are there any restrictions on the values of x for which this result is valid?

Preparing Your Lab Report

As before, your report should consist of a cover page with the title of the report and the names and signatures of your lab group members. **Also indicate your Lab Section.** Each person (in a group of three) should draft one of the three parts, and the group should meet to read and comment on these drafts before submitting the final report.

Part 1

A function $f(x)$ with formula $\cos(x)$ can have an inverse provided the domain is specified properly. The standard interval chosen near the origin is $0 \leq x \leq \pi$. Explain why a function with this formula could not have an inverse if the domain were extended to a larger interval. Then graph both $f(x)$ and its inverse function. This inverse function is named either $\cos^{-1}$ or arccos. Following the method from class (using the Chain Rule), find a formula for the derivative of the function $\arccos(x)$. You may need to use a trig identity to simplify your result.

Part 2

Suppose you know the derivative of an invertible function $f(x)$. Review the result relating the *derivatives* of f and f^{-1}. Under what conditions on f' will the *derivative* of f^{-1} become zero, become infinite, or fail to exist at all. You may wish to refer to the examples in this lab.

Part 3

In a manner similar to the last section of the lab, but starting with a right triangle whose hypoteneuse is x and whose "adjacent" side is a, derive relationships useful for finding the following antiderivative:

$$\int \frac{dx}{\sqrt{x^2 - a^2}}$$

Note: Ultimately you will be able to use the following result:

$$\int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + C$$

Check that your final antiderivative (expressed in terms of x) is correct by differentiating! (You can use *Derive* to help you or to check your work.) Are there any restrictions on the values of x for which this result is valid?