Consider the graph of $f(x) = \sqrt{4 - x^2}$ for $x \in [0, 2]$ below.

1. Setup. **TABLE WORK**

Recall that this curve is simply one quarter of the circle with radius $r = 2$ centered at $(0, 0)$. In the first lab, we saw that the area in the first quadrant under this curve was equal to π. Why is the *length* of the curve above also equal to π?

We can calculate lengths of curves by successive approximations using the distance formula. If we partition the interval $[0, 2]$ into N equal pieces, each with length Δx, and corresponding change in output value $\Delta y_k = f(t_{k-1}) - f(t_k)$, then

$$\text{LENGTH} \approx \sum_{k=1}^{N} \sqrt{(\Delta x)^2 + (\Delta y_k)^2}.$$

In order to verify this to the satisfaction of everyone in the group, on the picture above, break up the interval $[0, 2]$ into four pieces, and draw the four straight secant lines on the graph of the quarter circle you would use to approximate the length of the curve. Then write out and compute the sum $\sum_{k=1}^{4} \sqrt{(\Delta x)^2 + (\Delta y_k)^2}$.

2. **COMPUTER WORK**

The program **LENGTH.TRU** calculates the approximation on the previous page. Find this approximation using the program. You must adjust the program by defining the function f, giving the domain of the function, and telling it to use $n = 4$ steps.

Now find the length of $f(x) = \sqrt{4 - x^2}$ on the interval $[0, 2]$ accurate to four decimal places. You will need to adjust the number of steps, n, until you converge to three decimal places of accuracy. Give your approximate length as well as how many steps you used to get this. (Note: Organizing your work into a table may help you see the convergence to four decimal places.)

3. **TABLE WORK**

Converting to a Riemann Sum. The sum $\sum_{k=1}^{N}\sqrt{(\Delta y_k)^2+(\Delta x_k)^2}$ is NOT a Riemann sum. Do you know why?

We will now "convert" the sum $\sum_{k=1}^{N}\sqrt{(\Delta y_k)^2+(\Delta x_k)^2}$ into a Riemann sum. In fact, we will use the microscope equation $\Delta y \approx f'(a)\Delta x$ to do so!

The Microscope Equation

Consider a function $f(x)$ and the point **A** $(a, (f(a))$ on the graph. If we want to know the function value at some point **B** (b, y_b), *near* A, as long as the function is differentiable (locally linear) at A we can assume that the function can be approximated by the tangent at **A** and thus get an approximation for the exact value of the function at b in terms of a, $f(a)$ and b can be obtained by evaluating the tangent line at $x = b$. We'll call this approximation for $f(b)$ the value y_b

Equation of tangent line at $(a, f(a))$ is:

$$y = f(a) + f'(a)(x - a)$$

Value of the tangent line at $x = b$ is:

$$f(b) = f(a) + f'(a)(b - a)$$

But

$$y_b \approx f(b)$$

so,

$$y_b - f(a) \approx f'(a)(b - a)$$

If we think of $(b - a)$ as a change in x, i.e. Δx and $y_b - f(a)$ as a change in y, i.e. Δy we get what is known as the microscope equation:

$$\Delta y \approx f'(a)\Delta x$$

Fill in all the missing pieces below:

$$\text{LENGTH} \approx \sum_{k=1}^{N}\sqrt{(\Delta x_k)^2+(\Delta y_k)^2}$$

• Since the microscope equation $\Delta y \approx f'(a)\Delta x$ tells us that for small Δx, the corresponding Δy is approximately equal to $f'(a)\Delta x$, we shall replace Δy_k with $f'(x_k)\Delta x$. Note that x_k is our choice of sampling point. Do so below:

$$\text{LENGTH} \approx \sum_{k=1}^{N} \sqrt{(\Delta x)^2 + (\qquad\qquad)^2}$$

• Factor $(\Delta x)^2$ out of the radical. Be careful to see you are removing it from both terms and be careful about what goes outside the radical now.

$$\text{LENGTH} \approx \sum_{k=1}^{N} \sqrt{(\quad) + (\qquad\quad)^2} \quad (\qquad\quad)$$

• You **should** now have a Riemann sum, $\sum_{k=1}^{N} g(x_k)\Delta x$, where the function we are summing is

$$g(x) = \sqrt{(\qquad\qquad\qquad)},$$

where f is the function, the arclength of which we want to find. Explain why we have converted it into the form of a Riemann sum. **REMARK:** Be sure you understand that this is a Riemann sum using g, **NOT** f itself.

We conclude that

$$\text{Length} = \lim_{N\to\infty} \sum_{k=1}^{N} \sqrt{1 + [f'(x_k)]^2}\Delta x = \int_a^b \sqrt{1 + [f'(x)]^2}\ dx.$$

Before moving back to the computer, let's set up the Riemann sum which we obtained back in part 3 to find the length of $f(x) = \sqrt{4 - x^2}$ on the interval $[0, 2]$.

$f'(x) =$

$g(x) = \sqrt{[f'(x)]^2 + 1} =$

Simplify the function $g(x)$ above (using algebra) as much as possible before putting the function in the Riemann sum below:

$$\text{LENGTH} \approx \sum_{k=1}^{N} g(x_k)\Delta x_k =$$

4. COMPUTER WORK

(a) Use **RIEMANN.TRU** with the appropriate function, $g(x)$ above, the appropriate interval, and the left endpoint sample points, $x_k = a + (k - 1)\Delta x$ (or as the computer has it, LET x = a + k*Delta_x - (1)*Delta_x). Calculate this length to three decimal places of accuracy. What do you get as the length and how many subintervals did this require? Why do you think it required such a fine partition (so many small steps)? (Hint: Look at the function you're taking a Riemann sum over (NOT f!).)

(b) Try running the program with sample points on the right of each interval, $x_k = a + k\Delta x$, and try to obtain the length to three decimal places. What happens here and why did it happen?

(c) Try running the program with sample points *in the middle* of each interval, $x_k = a + k\Delta x - 0.5\Delta x$, and try to obtain the length to three decimal places.

(d) Think of an explanation for the difference in results you get when you use different sample points.