Discovering the Fundamental Theorem of Calculus

In this lab we will investigate the similarities and differences between Euler's method, which you learned last semester, and Riemann sums. Recall that for an initial value problem,

$$y'(x) = f(x, y) \qquad y(a) = b$$

Euler's method approximates the value of a solution $y(t)$ at a specific point t. Riemann sums on the other hand are used to estimate values of definite integrals such as,

$$\int_a^b g(t)dt.$$

Solving an Initial Value Problem Using Euler's Method

Use the True BASIC program EULER1, found under the `True Basic Calculus` icon (under the `Mathematics` icon), to obtain estimates to the solution of the following initial value problem. You will recall from last semester that EULER1 implements Euler's Method for an initial value problem with a single rate equation.

$$y'(x) = \sqrt{4 - x^2} \qquad y(-2) = 0.$$

To use this program you will need to specify the differential equation, initial values, and number of steps. Recall that the square root function in True BASIC is written $SQR(\cdot)$. You will also need to give high and low y values for the plot displaying the graph of the approximate solution Euler's Method calculates.

Use 16 steps. Record the integer part and the first 7 digits after the decimal place for each of the following estimates.

x	y(x)
-2	0
-1.5	
-1	
-0.5	
0	
0.5	
1	
1.5	
2	

Also record the plot of the *piecewise linear approximation* to the solution of this initial value problem. (This plot is given to you by EULER1, but you can also plot it yourself from the table.)

An Accumulation Function

Now consider the accumulation function

$$A(x) = \int_{-2}^{x} \sqrt{4 - t^2}\, dt \qquad .$$

On a separate piece of paper, draw a graph of the *integrand* $g(t) = \sqrt{4 - t^2}$. On what interval is this function defined? Given your answer to this question, on what interval is the accumulation function $A(x)$ defined?

Now use the True BASIC program AGGSUM to obtain right and left hand sum estimates for this accumulation function $A(x)$.

Choose $N = 16$, setting $a = -2$ and $b = 2$ in each case. AGGSUM is set up to calculate right hand sums. To use it to calculate left hand sums, you will need to make the following changes is the main loop for the program.

Change `FOR K = 1 TO N` *to* `FOR K = 0 TO N-1`.

Change `PRINT K, X, F(X), DELTA_AGG, AGG` *to*

`PRINT K, X + DELTA_X, F(X), DELTA_AGG, AGG`.

Again record the integer part and the first 7 digits after the decimal place for each estimate.

x	LHS	RHS
-2	0	0
-1.5		
-1		
-0.5		
0		
0.5		
1		
1.5		
2		

Compare the left hand sum estimates above with your results from solving the initial value problem using EULER1. What do you notice?

Be Able to Explain ...

Although no report is required for this lab, you should be able to explain the connection between Euler's Method for a particular initial value problem and one type of Riemann sum for a certain accumulation function. Pay particular attention to your reading in the *Calculus in Context* handouts. You may want to begin by writing down what Euler's Method is (get the others in your lab group to help). Then write down the Riemann sums, and go from there.

A Challenge

Now obtain new estimates which are **accurate to 1 decimal place**. **Warning:** $g(t) = \sqrt{4 - t^2}$ is NOT monotonic on the interval $-2 < t < 2$. Use a graphing calculator to graph $g(t)$ and help you plan your work. Record your estimates and error bounds below.

x	LHS	RHS	Estimate	± Error
-2	0	0	0	0
-1.5				
-1				
-0.5				
0				
0.5				
1				
1.5				
2				

An Extra Challenge

Consider the following three initial value problems:

$v'(t) = 2\cos(t^2)$ $\quad w'(t) = \cos(t^2) - \sin(t^2)$ $\quad z'(t) = \cos(t^2) + \sin(t^2)$
$v(-2) = 0$ $\quad w(-2) = 3$ $\quad z(-2) = -3$

Use the True BASIC program EULER1 to estimate solutions to these three initial value problems at the points indicated below. Try $N = 32$ in each case.

t	v(t)	w(t)	z(t)
-2	0	3	-3
-1.5			
-1			
-0.5			
0			
0.5			
1			
1.5			
2			

By writing the solutions to the last three initial value problems as accumulation functions, and referring to properties of integrals, show that $v(t) = w(t) + z(t)$. Did you observe this with your Euler approximations of these solutions? Discuss.