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Using Riemann Sums

Introduction: Riemann Sums

Riemann sums, named after the German mathematician G.F.B. Riemann (1826–1866), are used to express, in a consistent and systematic way, the results of the "subdivide, approximate, accumulate" approach to estimation of various quantities. Depending on the interpretation of a numerical function (as velocity, height, cross-section, power, etc.), the Riemann sum corresponds to an estimate of a related aggregated quantity (such as distance, area, volume, energy, etc.)

A particular Riemann sum for a function $f(x)$ defined on the interval $a \leq x \leq b$ is determined by a number of choices:

- N, the number of subdivisions of the interval;
- the lengths of each of the subintervals

  $$\Delta x_1, \Delta x_2, \ldots, \Delta x_k, \ldots, \Delta x_N$$

  (In some cases, all the subdivisions are of the same length, denoted by just Δx.); and
- the sequence of sampling points, one chosen from each of the subintervals:

  $$x_1, x_2, \ldots, x_k, \ldots, x_N.$$

The **Riemann sum** for the function f on the interval $[a, b]$ with this subdivision and this set of sampling points is defined to be:

$$\begin{aligned} \text{SUM} &= f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + \ldots + f(x_k)\Delta x_k + \ldots + f(x_N)\Delta x_N \\ &= \sum_{k=1}^{N} f(x_k)\Delta x_k \end{aligned}$$

A. Equal subdivisions

In the case when all the subdivision points are the same, the Riemann sum has the simpler form

$$\begin{aligned} \text{SUM} &= \sum_{k=1}^{N} f(x_k)\Delta x \\ &= \end{aligned}$$

1. Write out the "long-hand" version of the sum in the blank space above.
2. If the function is defined on the interval $[1, 6]$ and $N = 500$ equal subintervals are used, what is the value of Δx ?

   What number is at the right end of the first subinterval?

   What number is at the right end of the 17-th subinterval?

   What number is at the right end of the 217-th subinterval?

   What number is at the right end of the k-th subinterval? (For the last question, write a formula using the variable k.)

3. Write a formula which gives the value of Δx when N equal subintervals are used on the interval $[a, b]$.

   What number is at the right end of the first subinterval? (Write a formula using a and Δx.)

   What number is at the right end of the 217-th subinterval?

   What number is at the right end of the k-th subinterval?

When we wish to approximate some quantity using Riemann sums, we usually create a subdivision, pick our sampling points (usually with some scheme in mind – always the highest point, the lowest point, the midpoint, the right-hand endpoint, etc.), and make an approximation. To get a better approximation, refine the method by using more subintervals.

In today's lab, we will consider different functions on various intervals. Recall, we can interpret each Riemann sum as an approximation to the *signed area* between the interval on the x-axis and the graph of the function.

B. A monomial function

The first example is the monomial function $m(x) = \frac{2}{5}x^3$ for x in the interval $[1, 2]$. We will use Riemann sums to estimate the area under the graph $y = m(x)$, above the x-axis, for x between the left boundary $x=1$ and the right boundary $x=2$.

1. Sketch the graph $y = m(x) = \frac{2}{5}x^3, 1 \leq x \leq 2$ on a piece of graph paper. Of course, you may use a calculator or a computer to help.
2. Use $N=5$ subdivisions, so that $\Delta x = 0.2$. For this Riemann sum, use the *right-hand endpoints* as the sampling points $x_1, x_2, \ldots x_5$. Fill in the following table, using your calculators.
3. When you are complete, use the table to calculate the value of the Riemann sum.

$$\sum_{k=1}^{5} f(x_k)\Delta x$$

Identify on the table the five numbers you added to obtain the Riemann sum.

k	k-th interval	Right endpoint x_k	Value $f(x_k)$	$f(x_k) \cdot \Delta x$
1				
2				
3				
4				
5				

4. Start up the TRUE BASIC system on your group's computer and open the program file `AGGSUM`. This program will calculate a table just like you computed yourself, once you have filled in the following information:

   - the definition of the function $f(x)$,
   - the interval $[a, b]$, and
   - the number of subdivisions N.

   Type in this information and check your calculations above.

5. **More subdivision.** Keeping the function definition and the interval the same, increase the number of subdivisions from $N=5$ to $N=100$. What is the value of the Riemann sum with the refinement?

   Do you believe that this refined Riemann sum is closer to the value of the area? Express the basis for your belief in one or two sentences.

6. **Still more subdivision.** Use a large value of N for the number of subdivisions until you are confident that you have a "3-place" estimate for the exact value of the area. Explain the basis of your confidence. (If you are *sure* you have the best 3-place estimate, say why.)

   Note: A number EST with three digits after the decimal point is called the "three-place estimate" to the value VAL if EST is the result of "rounding" VAL to three places. One property of the three-place estimate is that $|EST - VAL| \leq \frac{1}{2}10^{-3}$

C. The reciprocal function

To calculate Riemann sums for the rest of this lab session, use the TRUE BASIC program `RIEMANN`. When you look at the program instructions, you will see that the calculations are all the same—the only difference is that the intermediate calculations are not displayed, only the summary.

For this set of problems, use the function $f(x) = \frac{1}{x}$.

1. On a sheet of plain paper, give a sketch of this function for $1 \le x \le 10$. Choose your scale to fill up as much of the page as possible with the sketch.
2. Interval $[1, 2]$. Use program `RIEMANN` and well-chosen values of N to give a two-place estimate of the area bounded by the x-axis and the graph $y = \frac{1}{x}$ above this interval.
3. Interval $[1, 3]$. Use program `RIEMANN` and well-chosen values of N to give a two-place estimate of the area bounded by the x-axis and the graph $y = \frac{1}{x}$ above this interval.

Since this exercise looks at areas bounded by this function over intervals $[1, b]$ for various values of b, it is convenient to define a new function as follows:

$$L(b) = \text{the area bounded by the } x\text{-axis and the graph } y = \frac{1}{x} \text{ above the interval } [1, b].$$

It is important to be aware that the *variable* in this newly-defined function is the *right boundary* b. Also, the geometric definition of the function does not apply for $b < 1$.

1. Use the program `RIEMANN` to calculate a two-place estimate of the number $L(6)$.
2. Complete the following table of values for the function L. Give two-place estimates for each of the values.

b	1	2	3	6		
$L(b)$						

3. Think up a special value of b for your group to use. Use the program `RIEMANN` to calculate two-place estimates of $L(b)$ and $L(2b)$. Fill in your values on the table above.
4. **A minor miracle.** Calculate the two differences $L(6) - L(3)$ and (using your group's value of b) $L(2b) - L(b)$. How do the differences compare? Can you interpret the result in terms of areas?

D. Estimating π

The number π is used in the formula for the area inside a circle: A circle with radius r has area $A = \pi r^2$. Students are commanded to memorize the value of the special number π, but are often given no reason to believe that the value is correct. This problem attempts to give a three-place estimate for π that every student can believe.

1. Draw the circle in the xy-coordinate plan with center at the origin and radius 2. Explain why the area inside the portion of the circle in the first quadrant is exactly π.

2. The equation for the entire circle is known to be $x^2 + y^2 = 4$. Use this equation to determine the function $y = f(x)$ whose graph is the portion of the circle in the first quadrant.

3. Use the computer—with appropriately specified function, interval, and number of subdivisions—to give a three-place estimate to π.

4. Is this method of approximating π more believable than memorizing what your middle-school math teacher told you?

Writing up this Lab.

You should complete the lab above and make sure that everyone in the group understands it. Later in the week, your lab group should meet once again to discuss what you learned from this lab. As a group, prepare your responses to the questions posed above in Part B, C and D, after checking your work in Part A. Your report should consist of a coherent narrative for each of Parts B, C, and D; in each case, clearly state the problem to be solved and answer each question in that part in the course of your narrative. Include any necessary and supporting graphs, mathematical reasoning and calculation. Reports may be word-processed, but neat and clear hand written reports are perfectly acceptable.

This summary is due at lab time in one week **(Wednesday/Thursday, February 7/8, 2001)**. Each member of the group should sign her/his name on the submitted paper, indicating her/his full participation in the preparation of the assignment.