# Math 120 Spring 2001 Final Review Assignment and Topics Summary

**Ron Buckmire**
**Alan Knoerr**

0. **The Final Exam in Math 120 is Thursday May 10 6:30pm-9:30pm in Fowler**

1. **The ideas are the most important thing!**

2. **Practice some techniques.** Important techniques include determining the number of subdivisions needed to obtain a Riemann sum approximation of the definite integral of a monotone function to a given degree of accuracy; finding the derivative of an accumulation function; relating the graphs of a function, its derivative and its family of antiderivatives; writing the solution of an initial value problem as an accumulation function; using basic properties of integrals and antiderivatives; using the Fundamental Theorem of Calculus to evaluate definite integrals; approximating the value of a definite integral by approximating the integrand with a Taylor Series. In particular, you should know the table of antiderivatives/derivatives below.

a.

| $f'(x) = F''(x)$ | $f(x) = F'(x)$ | $F(x) = \int f(x)dx$ |
|---|---|---|
| $0$ | $1$ | $x + C$ |
| $nx^{n-1}$ | $x^n \ (n \neq -1)$ | $\dfrac{1}{n+1}x^{n+1} + C$ |
| $-\frac{1}{x^2}$ | $\frac{1}{x}$ | $\ln\|x\| + C$ |
| $\cos(x)$ | $\sin(x)$ | $-\cos(x) + C$ |
| $-\sin(x)$ | $\cos(x)$ | $\sin(x) + C$ |
| $2\sec^2(x)\tan(x)$ | $\sec^2(x)$ | $\tan(x) + C$ |
| $-\dfrac{1}{(1+x^2)^2} \cdot 2x$ | $\dfrac{1}{1+x^2}$ | $\arctan(x) + C$ |
| $\frac{1}{2}(1-x^2)^{-3/2} \cdot 2x$ | $\frac{1}{\sqrt{1-x^2}}$ | $\arcsin(x) + C$ |
| $e^x$ | $e^x$ | $e^x + C$ |
| $a^x \ln(a)$ | $a^x (a > 0)$ | $\frac{1}{\ln(a)}a^x + C$ |
| $\sec^2(x)$ | $\tan(x)$ | $-\ln(\|\cos(x)\|) + C$ |
| $\dfrac{1}{x}$ | $\ln(x)$ | $x\ln(x) - x + C$ |

b. finding antiderivatives and evaluating definite integrals by using $u$-substitution ($\int f'[g(x)]g'(x)\,dx = \int f'[u]\ du = f[u] + C = f[g(x)] + C$ pick $u = g(x)$ and convert the integral entirely to $u$ variables. You can either return to $x$-variables or stay in $u$-space when evaluating definite integrals); However you should be able to convert one definite integral in $x$-variables entirely to another integral in $u$-variables given the particular $u$-substitution

c. finding antiderivatives and evaluating definite integrals by using integration by parts ($\int f \cdot g'\,dx = f \cdot g - \int f' \cdot g\,dx$ or $\int u dv = uv - \int v du$). Don't forget

**(i)** Repeated integration by parts e.g. $\int x^2 e^{-x} dx$

d. finding the area between two curves by setting up the proper integral over the proper interval and evaluating it using the fundamental theorem of calculus or approximating it using numerical methods;

e. finding the average value of a function on a given interval by setting up the proper integral over the proper interval, i.e. $\frac{1}{b-a}\int_a^b f(x)dx$

f. finding the length of a curved segment of $f(x)$ from $(a, f(a))$ to $(b, f(b))$ by using the formula $L = \int_a^b \sqrt{1 + [f'(x)]^2} dx$

g. using basic algebra skills and trigonometric identities to help simplify integrands, so that one may find antiderivatives and evaluate definite integrals;

h. using numerical methods of integration:

- Riemann Sums $\sum_{k=1}^{N} \mathtt{f(x_k)\Delta x}$
- Left Hand Sums $\mathtt{x_k = a + (k-1) * \Delta x}$, Right Hand Sums $\mathtt{x_k = a + k * \Delta x}$
- Midpoint method $\mathtt{x_k = a + (k - .5) * \Delta x}$
- Trapezoidal Rule $T = \frac{L+R}{2}$
- Simpson's Rule $S = \frac{2}{3}M + \frac{1}{3}T$
- Midpoint Error versus Trapezoid Error (depends on concavity $f''$ and is proportional to $N^{-2}$ or $(\Delta x)^2$)
- Left Riemann Error versus Right Riemann Error (depends on slope $f'$ and is proportional to $N^{-1}$ or $\Delta x$)
- Simpson Error (depends on $f^{(4)}$ and is proportional to $N^{-4}$ or $(\Delta x)^4$)
- Error Control e.g. Riemann error $= |f(b) - f(a)|\dfrac{b-a}{N} \leq .001 \Rightarrow$ solve for $N$

3. Other topics include:

   a. improper integrals of the first kind and of the second kind (remember the $p$-rules!)

   b. determining convergence of improper integrals using the Comparison Test for Improper Integrals

   c. polynomial approximations of a function near a point (Taylor polynomials), applications of Taylor polynomial approximations to derivatives and anti-derivatives, using calculus and algebra to find new Taylor Series from familiar ones

   d. tests for DIVERGENCE of an infinite series **n-th term a.k.a.(zero-limit); tests for convergence and/or divergence: alternating series, integral, comparison, absolute ratio and geometric series**

e. useful series to remember are $p$-series, geometric series, harmonic series, alternating harmonic series. Remember that the sum of a geometric series $\sum_{k=0}^{\infty} ar^k$ converges to $\frac{a}{1-r}$ as long as the ratio $|r| < 1$

f. Remember the differences and connections between improper integrals and infinite series and be able to articulate these concepts in written form

4. **Inverse Functions**. Recall the basic definition of an inverse. It is a function $g(x)$ which is related to another function $f(x)$ such that $f(g(x)) = x$ and $g(f(x)) = x$.

   a. Recall that the graph of a inverse of a function can be obtained from the original function by reflection across the line $y = x$.

   b. Recall also that the derivative $g'(b) = \frac{1}{f'(a)}$ when $b = f(a)$ and $a = g(b)$ when $f$ and $g$ are inverses of each other.

5. **Practice using tests for convergence.** Especially important are the Absolute Ratio Test and the Zero Limit Test for Divergence. Don't come into the exam without being able to take the limit as $k \to \infty$ of some expression involving $k$. You should be able to apply L'Hopital's rule on those indeterminate limits. Don't forget the other tests we have covered (the Integral Test, the Comparison Test, and Alternating Series Test).

6. Remember the basic idea of doing comparisons:
   If you want to show that something CONVERGES, you have to compare it to something which is **LESS THAN OR EQUAL TO** something you already know CONVERGES.
   If you want to show that something DIVERGES, you have to compare it to something which is **GREATER THAN OR EQUAL TO** something you already know DIVERGES.
   The "something" can either be an improper integral or an infinite series, but in either case the integrand or terms must all be POSITIVE. FUNCTIONS DO NOT converge or diverge, improper integrals or infinite series do.

7. Taylor Series To Remember...

$$\begin{aligned}
\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots &&= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \\
\cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots &&= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \\
\tfrac{1}{1-x} &= 1 + x + x^2 + x^3 + \dots &&= \sum_{k=0}^{\infty} x^k \\
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} + \dots &&= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} \\
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots &&= \sum_{k=0}^{\infty} \frac{x^k}{k!}
\end{aligned}$$

7. **Evaluating Limits**

L'Hôpital's Rule

If $\lim_{x\to\infty} \dfrac{f(x)}{g(x)}$ is of the form $\frac{\infty}{\infty}$ or $\frac{0}{0}$ or $0 \cdot \infty$ then

if if the limit $\lim_{x\to\infty} \dfrac{f'(x)}{g'(x)} = L$ exists, then $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = L$ In other words, if you have an indeterminate limit, just differentiate the numerator and denominator and take the limit again until you get a determinate answer. That answer will be the value of the limit.

You should be comfortable with discounting or ignoring parts of an expression when these parts are getting very small compared to the rest of the expression. Remember, $\sin(x)$ and $\cos(x)$ only return values between $\pm 1$.

Recall the various rules involving limits, such as $\lim_{b\to\infty} b^p$, $\lim_{x\to\infty} e^{kx}$

Remember that when there's a race between $e^x$ and any polynomial function $x^p$ as $x \to \infty$, $e^x$ will always win. Conversely, $\ln(x)$ will lose any race with $x^p$ as $x \to \infty$

8. **Interval Of Convergence and Radius Of Convergence**

Consider $\sum_{k=0}^{\infty} b_k (x-a)^k$. This Power Series may not converge for all $x$-values. The set of $x$-values for which the series converges is called the *interval of convergence*. The interval of convergence is always centered on the point $a$.

The interval of convergence can be infinite, i.e. $(-\infty, \infty)$ a.k.a. "all Real Numbers". Or it can be a finite interval of the form $(a-R, a+R)$, $[a-R, a+R]$, $(a-R, a+R]$ or $[a-R, a+R)$. $R$ is called the *radius of convergence*. The value $R = 1/L$ and is computed using the absolute ratio test.

$$\frac{1}{R} = L = \lim_{k\to\infty} \left| \frac{b_{k+1}}{b_k} \right|$$

We use this idea of interval of convergence in the SPECIFIC EXAMPLE of trying to determine for which $x$ values a Taylor Series will converge. (Remember a Taylor Series is just a special case of Power Series.)

# FINAL EXAM BLUE NOTES

Name: ______________________________

## SAMPLE PROBLEMS FOR FINAL EXAM

1. Draw the area represented by the following integral

$$\int_0^4 \frac{1}{1+x^2}\,dx.$$

Using $n = 4$ subintervals, estimate the definite integral using the following:

a. left endpoint Riemann sum

b. right endpoint Riemann sum

c. midpoint Riemann sum

d. trapezoid rule and

e. Simpson's rule.

f. Compute the exact integral (use the FTC).

Then compare your estimates. Know which ones are most accurate and why.

2. a. Using integration by substitution, find $\int x\sqrt{1+3x}\,dx$

b. Using integration by parts, find $\int \frac{\ln(x)}{x^2}\,dx$

c. Using integration by parts or integration by substitution, find $\int \sin^2(x)\cos x\,dx$

d. By evaluating a definite integral, find the area under the $x$-axis but above the curve $y = x^2 - 3x$. Draw a sketch of the curve and indicate the requested area on your sketch.

e. Below is a list of indefinite integrals. Find an antiderivative for each.

$$\int \sin(2x)dx, \quad \int 3\cos^2(4x)\sin(4x)\,dx, \quad \int 7^x\,dx, \int x^2(x^3-6)^{20}\,dx, \quad \int \frac{3}{4x-2}\,dx.$$

f. Find the average value of $f(x) = \sin^2(3x)\cos(3x)$ on $[0, \frac{\pi}{6}]$.

g. Evaluate the following:

$$\int_0^1 \frac{x}{\sqrt{1+x}}\,dx, \quad \int \frac{x}{1+\sqrt{x}}\,dx.$$

3. Explain why using Simpson's method to evaluate the definite integral in part (d) above will compute the answer **exactly**, but if you were to use the Midpoint or Trapezoid Method the answer would only be approximate. (You can test this for yourself by trying to evaluate the definite integral using Midpoint, Trapezoid and Simpson's Method and seeing that Simpson's is exact.)

4. Calculate the following (improper) integrals.

   a. $\displaystyle\int_0^1 \ln(2x)\,dx$

   b. $\displaystyle\int_0^6 \frac{1}{\sqrt{6-x}}\,dx$

   c. $\displaystyle\int_3^8 \frac{1}{2-x}\,dx$

   d. $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}+2}\,dx$

   e. $\displaystyle\int_1^\infty \frac{x^2}{1+\sin^2(x)}dx$

   f. $\displaystyle\int_1^\infty \frac{\sin^2(x)}{1+x^2}dx$

5. Calculate the area of the region under the curve $y = \sqrt{x}+1$, above the $x$-axis and between $x = 0$ and $x = 4$.

6. Find the unique solution to the IVP

$$y' + y(1+x) = 0, \qquad y(-2) = 1$$

7. Given that $A(\mathcal{X}) = \int_1^{\mathcal{X}} \sqrt{1+e^{2x}}\,dx$

   a. Evaluate $A(1)$
   b. Evaluate $A'(1)$
   c. Evaluate $A''(1)$
   d. Show that $A(b)$ represents the length of the curve $y = e^x$ from the coordinate $(1, e)$ to $(b, e^b)$
   e. If $B(\mathcal{X}) = \int_1^{\sin(\mathcal{X})} \sqrt{1+e^{2x}}\,dx$, Find $B'(\mathcal{X})$.

8. Compute the following:
   $A = \int_0^2 x^2 t\,dx$
   $B = \int_0^2 x^2 t\,dt$
   $C = \int_0^2 x^2 t\,dk$
   $D = \int_0^x k^2 t\,dt$
   $E = \int_0^k x^2 t\,dx$
   What is $\frac{dB}{dx}$ equal to? What about $\frac{dD}{dx}$? $\frac{dE}{dx}$? $\frac{dA}{dt}$?

9. Solve the following initial value problem

$$f'(x) = \frac{3}{x} + e^x + x^3, \qquad f(2) = 0.$$

10. a. Find the second degree Taylor polynomial based at $a = 2$ for the function $g(x) = e^{2-x}$.

    b. Use your answer in part (a) to estimate

$$\int_{1.9}^{2.1} e^{2-x}\,dx.$$

11. Determine whether the following series converge or diverge.

a. $\sum_{n=0}^{\infty} \frac{1}{3+4^n}$

b. $\sum_{n=0}^{\infty} \frac{6^n}{n!}$

c. $\sum_{n=1}^{\infty} (-1)^n \frac{5}{\sqrt{n^3}}$

d. $\sum_{n=1}^{\infty} (-1) \frac{2}{\sqrt{n}}$

12. Evaluate the following sums exactly:

$$1 + \frac{1}{e} + \frac{1}{e^2} + \frac{1}{e^3} + \cdots .$$

$$1 - \frac{1}{e} + \frac{1}{e^2} - \frac{1}{e^3} + - \cdots .$$

$$1 + 2\frac{1}{e} + 3\frac{1}{e^2} + 4\frac{1}{e^3} + \cdots .$$

$$1 + \frac{1}{e^2} + \frac{1}{e^4} + \frac{1}{e^6} + \cdots .$$

14. Write down the Taylor Series for $f(x) = e^{-x^3}$ about the point $x = -1$. (HINT: You probably do **not** want to do this by taking any derivatives of $f(x)$.)

15. Find the radius and interval of convergence of the infinite series $\sum_{k=0}^{\infty} (-1)^k x^{2k}$

16. Write down an example of a function which is *involutory*. That is, it is its own inverse. You should be able to think of atleast one example and then using the definition of an inverse prove that the function satisfies this definition.