Definite Integrals

Many scientific problems can be solved by the following strategy:

**Subdivide-Approximate-Accumulate-Refine**

Examples we have studied include determining area, volume, distance traveled, human work done and electrical energy consumed. Let $Q$ be such a quantity. Often the subdivisions $\Delta Q_k$ can be approximated by terms of the form

$$\Delta Q_k \approx f(t_k) \Delta t_k, \quad a \leq t_0 < \cdots < t_n \leq b, \quad \Delta t_k = t_k - t_{k-1},$$

with $f$ at least piecewise continuous on the interval $[a, b]$. For volume, $f$ is the cross-sectional area as a function of distance from the end of the object; for distance traveled, $f$ is the velocity as a function of time; for human work done, $f$ is the staffing level as a function of time; and for electrical energy consumed, $f$ is the electrical power as a function of time.

Accumulating these subdivision approximations yields a **Riemann sum**. The sum is a left-hand sum (LHS) if $f$ is evaluated at the left-endpoint of each subinterval. It is a right-hand sum (RHS) if $f$ is evaluated at the right-endpoint of each subinterval. (If the function is evaluated at other points in the subinterval no special name is given to the Riemann sum.)

\[
\text{(LHS): } \sum_{k=0}^{n-1} f(t_k) \Delta t_k, \quad \text{(RHS): } \sum_{k=1}^{n} f(t_k) \Delta t_k, \quad a \leq t_0 < \cdots < t_n \leq b.
\]

If $f$ is piecewise continuous and/or monotone, refining the subdivisions so that $\Delta t_k \to 0$ as $n \to \infty$ causes these sums to converge to the same value, a **definite integral**:

\[
\lim_{n \to \infty} \sum_{k=0}^{n-1} f(t_k) \Delta t_k = \lim_{n \to \infty} \sum_{k=1}^{n} f(t_k) \Delta t_k = \int_{a}^{b} f(t) \, dt.
\]

A useful interpretation of this definite integral is as the signed area between the graph of $f(t)$ and the $t$ axis over the interval $[a, b]$. Area below the axis is regarded as negative in this interpretation.

$f$ is **monotone increasing** on $[a, b]$ if it does not decrease on this interval. In this case,

$$\text{LHS} \leq \int_{a}^{b} f(t) \, dt \leq \text{RHS} \quad \text{and} \quad |\text{RHS} - \text{LHS}| = |f(b) - f(a)| \cdot \Delta t, \quad \Delta t = (b - a)/n.$$

$f$ is **monotone decreasing** on $[a, b]$ if it does not increase on this interval. In this case,

$$\text{RHS} \leq \int_{a}^{b} f(t) \, dt \leq \text{LHS} \quad \text{and} \quad |\text{RHS} - \text{LHS}| = |f(b) - f(a)| \cdot \Delta t, \quad \Delta t = (b - a)/n.$$

Thus the **accuracy** of Riemann sum approximations to integrals of montone functions and the number $n$ of subdivisions needed to achieve a certain accuracy can be calculated.
Using both the definition and geometric interpretations, it is easy to establish the following properties of definite integrals:

\[ \int_a^b f(t) \, dt + \int_a^b g(t) \, dt = \int_a^b f(t) + g(t) \, dt, \quad \int_a^b c \cdot f(t) \, dt = c \cdot \int_a^b f(t) \, dt, \quad \text{constant} \, c, \]

\[ \int_a^b f(t) \, dt + \int_b^c f(t) \, dt = \int_a^c f(t) \, dt, \quad \int_a^b f(t) \, dt = -\int_b^a f(t) \, dt \]

**Accumulation Functions and the Fundamental Theorem of Calculus (FTC)**

A definite integral evaluates to a single number. However, if we allow the upper limit of the interval of integration to vary we get a function called an **accumulation function**:

\[ F(T) = \int_a^T f(x) \, dx. \]

Thinking about Riemann sum approximations to the integral with fixed \( T \), it is easy to see that \( F(T + \Delta T) - F(T) \approx f(T)\Delta T \). From this follows one version of the **FTC**:

\[ F'(T) = \frac{d}{dT} \left( \int_a^T f(x) \, dx \right) = f(T), \quad \text{if} \, f \, \text{is continuous at} \, T. \]

Observing that \( F(a) = 0 \), we have another version of the **FTC**:

\[ y(T) = \int_a^T f(t) \, dt \quad \text{is the solution of} \quad \begin{cases} y'(T) = f(T) \\ y(a) = 0 \end{cases} \]

provided \( f \) is continuous at \( T \).

This theorem is also illustrated by looking carefully at Euler’s method for this initial value problem and left-hand Riemann sums for the accumulation function. They are identical!

**Antiderivatives and the Fundamental Theorem of Calculus**

If \( F'(x) = f(x) \), then \( F \) is said to be an **antiderivative** of \( f \). The antiderivative of \( f \) is not unique, but two such antiderivatives differ only by an added constant. The last version of the **FTC** states that if \( F \) is an antiderivative of \( f \), then

\[ \int_a^b f(t) \, dt = F(b) - F(a). \]

An antiderivative of a continuous function \( f \) can always be written as an accumulation function. While this can always be approximately evaluated numerically, it is sometimes possible to find a formula for an antiderivative in closed form. In seeking closed forms remember the following: **Every differentiation rule implies an antiderivation rule**.

Thus the basic derivative formulas you know well provide basic antiderivative formulas. More complicated formulas are found in tables or through symbolic computing packages like Derive.