Taylor used polynomials to approximate functions.
Fourier used trigonometric functions to approximate periodic functions.

We write $P_n(x)$ for the $n$th degree Taylor polynomial.
Example: $P_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$.

We write $F_n(x)$ for the $n$th degree Fourier “polynomial.”
Example: $F_3(x) = a_0 + a_1 \cos(x) + a_2 \cos(2x) + a_3 \cos(3x) + b_1 \sin(x) + b_2 \sin(2x) + b_3 \sin(3x)$.
$a_k$ and $b_k$ are some constants. They are called the coefficients.

A Taylor Series is $\sum_{k=1}^{\infty} a_k x^k$. A Fourier Series is $a_0 + \sum_{k=1}^{\infty} b_k \sin(kx) + a_k \cos(kx)$

For a periodic function $f(t)$ whose period is $2\pi$, the coefficients of its Fourier Series are:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt$$
$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(kt) dt, \ k = 1, 2, 3, \cdots$$
$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(kt) dt, \ k = 1, 2, 3, \cdots$$

**EXAMPLE**

$$f(x) = \begin{cases} 
7 & \text{if } (2n)\pi \leq x \leq (2n + 1)\pi \\
0 & \text{if } (2n + 1)\pi < x < (2n + 2)\pi
\end{cases}$$

1. Sketch the graph of $f(x)$ below.

2. Find the first degree Fourier polynomial for $f(x)$.

3. Find the second degree Fourier polynomial for $f(x)$.
Fourier Series
In general, a Fourier Series is used to approximate a function \( f(t) \) with period \([0, T]\)

\[
f(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos\left(\frac{2k\pi}{T} t\right) + \sum_{k=1}^{\infty} b_k \sin\left(\frac{2k\pi}{T} t\right)
\]

where

\[
a_0 = \frac{1}{T} \int_0^T f(t) \, dt
\]
\[
a_k = \frac{2}{T} \int_0^T f(t) \cos\left(\frac{2k\pi}{T} t\right) \, dt
\]
\[
b_k = \frac{2}{T} \int_0^T f(t) \sin\left(\frac{2k\pi}{T} t\right) \, dt
\]

This usually involves a fair amount of integration.

**EXAMPLE**

\( f(x) = \begin{cases} 
    x & \text{if } -1 \leq x \leq 0 \\
    -x & \text{if } 0 < x < 1 
\end{cases} \)

1. Sketch the graph of \( f(x) \) below.

2. Find the zeroth degree Fourier polynomial for \( f(x) \).

3. How would you show the general form of the Fourier series for \( f(x) \) is \( F_\infty(x) = \frac{-1}{2} + \sum_{k=1}^{\infty} \frac{2 - 2 \cos(k\pi)}{(k\pi)^2} \cos(k\pi x) \).

4. For what values of \( x \) will the infinite series converge? Which test would you use?
Math 118 Fall 2002, Quiz 10. Consider the function $f(x)$ which has period $2\pi$

$$f(x) = \begin{cases} 
-1, & \text{when } -\pi < x < 0 \\
1, & \text{when } 0 \leq x < \pi 
\end{cases}$$

(a) (2 points) Sketch a graph of $f(x)$ on the interval from $-3\pi \leq x \leq 3\pi$ in the space below

(b) (4 points) Compute the value of the $a_k$ (cosine) coefficients of the Fourier series, where $k = 0, 1, 2, 3, \ldots$

(c) (4 points) Compute the value of the $b_k$ (sine) coefficients of the Fourier series, where $k = 1, 2, 3, \ldots$
The figures in the first column show $F_1(x)$, $F_7(x)$ and $F_{15}(x)$ which are the solution to Quiz 10, Fall 2002.

In the righthand column $F_1$, $F_3$ and $F_{11}$ are shown for the example on the previous page.