Class 35: Monday December 1

Introduction to Fourier Series

Taylor used polynomials to approximate functions.
Fourier used trigonometric functions to approximate **periodic** functions.

We write $P_n(x)$ for the nth degree Taylor polynomial.
Example: $P_3(x) = a_0 + a_1x + a_2x^2 + a_3x^3$.

We write $F_n(x)$ for the nth degree Fourier "polynomial."
Example: $F_3(x) = a_0 + a_1\cos(x) + a_2\cos(2x) + a_3\cos(3x) + b_1\sin(x) + b_2\sin(2x) + b_3\sin(3x)$.
a_k and b_k are some constants. They are called the coefficients.

A Taylor Series is $\sum_{k=1}^{\infty} a_kx^k$. A Fourier Series is $a_0 + \sum_{k=1}^{\infty} b_k\sin(kx) + a_k\cos(kx)$

For a periodic function $f(t)$ whose period is 2π, the coefficients of its Fourier Series are:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt \qquad a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt)dt,\ k = 1, 2, 3, \cdots$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt)dt,\ k = 1, 2, 3, \cdots$$

EXAMPLE

$$f(x) = \begin{cases} 7 & \text{if } (2n)\pi \le x \le (2n+1)\pi \\ 0 & \text{if } (2n+1)\pi < x < (2n+2)\pi \end{cases}$$

1. Sketch the graph of $f(x)$ below.

2. Find the first degree Fourier polynomial for $f(x)$.

3. Find the second degree Fourier polynomial for $f(x)$.

Fourier Series

In general, a Fourier Series is used to approximate a function $f(t)$ with period $[0, T]$

$$f(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(\frac{2k\pi}{T}t) + \sum_{k=1}^{\infty} b_k \sin(\frac{2k\pi}{T}t)$$

where

$$\begin{aligned} a_0 &= \frac{1}{T}\int_0^T f(t)\, dt \\ a_k &= \frac{2}{T}\int_0^T f(t)\, \cos(\frac{2k\pi}{T}t)\, dt \\ b_k &= \frac{2}{T}\int_0^T f(t)\, \sin(\frac{2k\pi}{T}t)\, dt \end{aligned}$$

This usually involves a fair amount of integration.

EXAMPLE

$f(x) = \begin{cases} x & \text{if } -1 \le x \le 0 \\ -x & \text{if } 0 < x < 1 \end{cases}$

1. Sketch the graph of $f(x)$ below.

2. Find the zeroth degree Fourier polynomial for $f(x)$.

3. How would you show the general form of the Fourier series for $f(x)$ is $F_\infty(x) = -\frac{1}{2} + \sum_{k=1}^{\infty} \frac{2 - 2\cos(k\pi)}{(k\pi)^2} \cos(k\pi x)$.

4. For what values of x will the infinite series converge? Which test would you use?

Math 118 Fall 2002, Quiz 10. Consider the function $f(x)$ which has period 2π

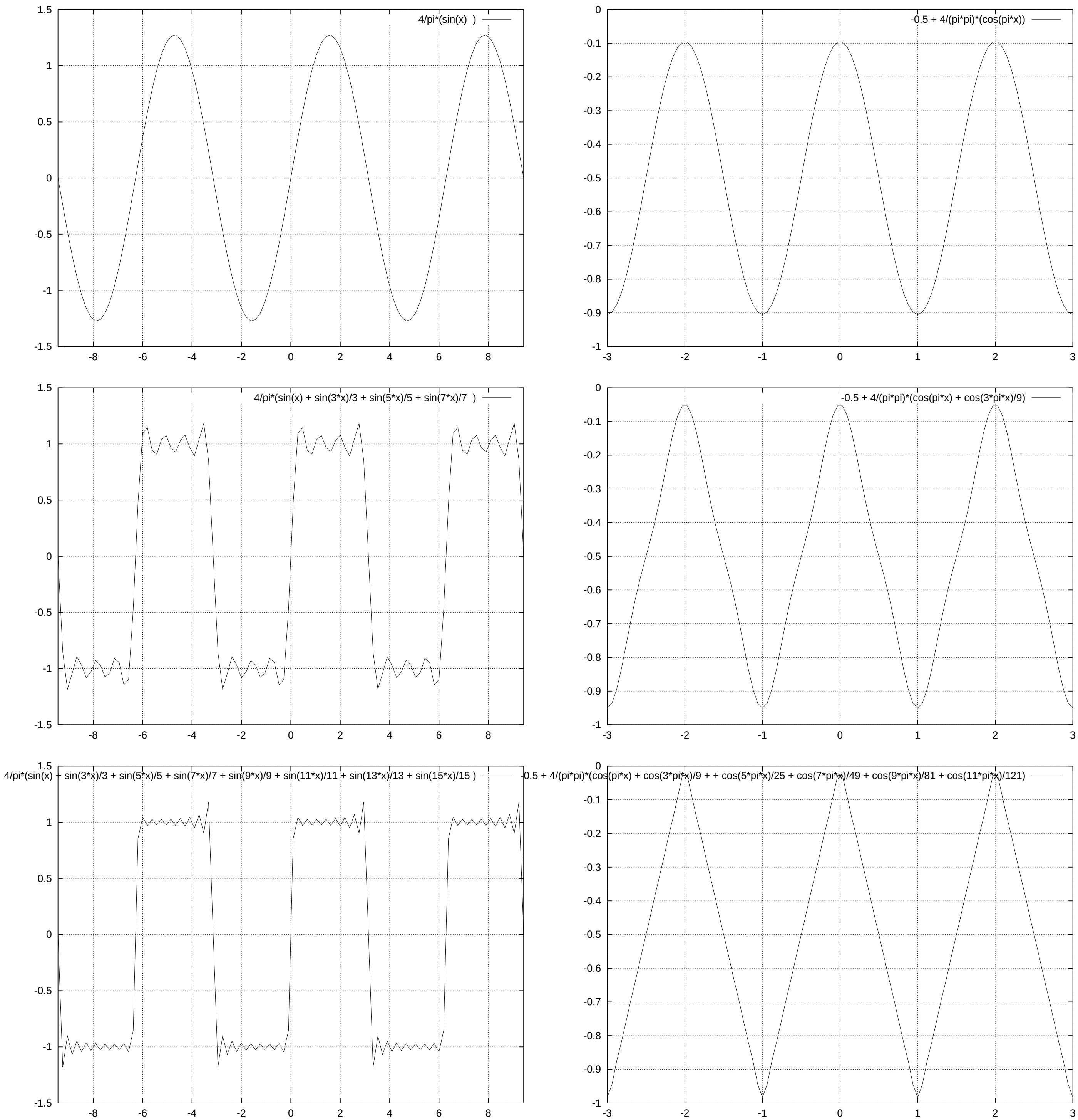
$$f(x) = \begin{cases} -1, & \text{when } -\pi < x < 0 \\ 1, & \text{when } 0 \leq x < \pi \end{cases}$$

(a) (*2 points*) Sketch a graph of $f(x)$ on the interval from $-3\pi \leq x \leq 3\pi$ in the space below

(b) (*4 points*) Compute the value of the a_k (cosine) coefficients of the Fourier series, where $k = 0, 1, 2, 3, \ldots$

(c) (*4 points*) Compute the value of the b_k (sine) coefficients of the Fourier series, where $k = 1, 2, 3, \ldots$

The figures **in the first column** show $F_1(x)$, $F_7(x)$ and $F_15(x)$ which are the solution to Quiz 10, Fall 2002.



In **the righthand column** F_1, F_3 and F_{11} are shown for the example on the previous page.