Class 29: Monday November 10

Application of Taylor Polynomials to IVPs

The general form of a Taylor Polynomial approximation to a function $f(x)$ about a point $(a, (f(a))$ is

$$P_n(x;a) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 + \ldots \frac{f^{(n)}}{n!}(x-a)^n = \sum_{k=0}^{n} \frac{f^{(k)}}{k!}(x-a)^k$$

which we could write in general as

$$P_n(x;a) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n = \sum_{k=0}^{n} a_k x^k$$

Infinite Series

A Taylor Series is a Taylor Polynomial of infinite degree. It looks like $P_\infty(x;0) = \sum_{k=0}^{\infty} a_k x^k$

A Taylor Series is a special form of an **infinite series**.

If the Taylor Series converges, then it represents the function *exactly*, in other words $f(x) = P_\infty(x;0)$

As you may recall, an infinite series can either converge (i.e. add up to a finite value) or diverge (become infinitely large or never get infinitely close to one single value).

EXAMPLES:

Which of the following infinite series converge? (write out the the terms of the series and see if you can see a trend in the partial sums)

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

$$\sum_{k=0}^{\infty} k$$

$$\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$$

Taylor Series and IVPs

We'll discuss more about infinite series in general in future classes, right now we will be showing how Taylor Series can be used as another technique for approximating solutions to initial value problems.

Suppose we have the IVP $y' = 2y, \quad y(0) = 3$. Even though we know what function solves this IVP (right?) we can approximate it by a Taylor Series.

Let $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$

We have an infinite number of unknowns, a_i, with $i = 0,\ 1,\ 2,\ 3$, etc

We can differentiate y with respect to x to obtain y'.

Using the initial condition, what equation can we obtain for a_i?

Using the differential equation, what equation can we obtain for a_i? This kind of equation is called a **recurrence relation**.

For $k = 0$ what is the recurrence relation?

For $k = 1$?

For $k = 2$?

For any k?

Thus the Taylor Series representation is:

Consider the IVP $y' = xy, \quad y(0) = 1$.

(a) Show that the Taylor Series for the solution of this IVP looks like $\sum_{k=0}^{\infty} \frac{x^{2k}}{2^k \cdot k!} = 1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{24} + \ldots$

(b) The solution of this IVP can also be obtained using separation of variables.

(c) Is the Taylor Series for the function you found the same series in part (b)?