Class 22: Friday October 24

Application of Integration

An Algorithm For Computing Antiderivatives and Definite Integrals

1. Is it an antiderivative or a definite integral (i.e. is the answer a family of functions or a number?)
2. Try to simplify the integrand.
3. Consult your table of antiderivatives.
4. Does the integrand consist of a product of functions?
5. Do you see a composite function in the integrand? Do you also see the derivative of the "inside function" multiplying the "dx"?
6. If using integration by substitution, make sure you can convert the ENTIRE integral into the new variable.
7. If using integration by parts, you should choose carefully which function you want to differentiate and which function you want to anti-differentiate.
8. If it is a definite integral, you can use numerical methods (Riemann sums) to approximate the answer.
9. If it is an antiderivative, you can also consult a table of integrals or a computer program like `Derive`.
10. **CHECK YOUR ANTIDERIVATIVE, BY DIFFERENTIATING IT TO PRODUCE THE INTEGRAND!**

Integration by Substitution

Let $h(x) = f(g(x))$. Then $h'(x) = f'(g(x)) \cdot g'(x)$ using **the Chain Rule for Differentiation.** Integration by Substitution involves the reverse of this process.

Consider $\mathcal{I} = \int f'(g(x))g'(x)dx$.
Let $u = g(x)$ We know $u' = g'(x)$ Therefore, $\mathcal{I} = \int f'(u)u'(x)dx = \int f'(u)du = f(u) + C = f(u(x)) + C$ since we know $u'(x) = \dfrac{du}{dx}$ and $u'(x)dx = du$.
In other words,

$$\int f'(u(x))u'(x)dx = f(u(x)) + C$$

Integration by Parts

Let $h(x) = u(x) \cdot v(x)$. Then $h'(x) = u'(x) \cdot v(x) + u(x) \cdot v'(x)$ using the **Product Rule for Differentiation**. Integration by Parts involves the anti-differentiation of this equation.

Anti-differentiate both sides of the product rule equation $[uv]' = u'v + uv']$

$$\begin{aligned} \int [u(x) \cdot v(x)]'dx &= \int u'(x) \cdot v(x) + u(x) \cdot v'(x)dx \\ u(x) \cdot v(x) &= \int u'(x) \cdot v(x)dx + \int u(x) \cdot v'(x)dx \end{aligned}$$

Therefore:

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx$$

Examples of Problems Which Involve Integration

Separation of Variables
Solve the initial value problem

$$y' = (1 + y^2)e^t, \qquad y(0) = 0$$

Area Between Curves
What is the area between the curves $x^{\sqrt{5}}$ and $\dfrac{2x}{1+x^2}$ in the first quadrant?

Average Value of a Function
What is the average value of the $\ln^2(x)$ between 1 and e?

Curve Length
Find the length of the curved line segment between $(0,0)$ and $(1,1)$ tracing the path $y^2 = x^3$.

My Favorite Test Question Involving Integration

Given the following information about an unknown function $g(x)$

$$\int_1^2 g(t)\,dt = 3, \quad \int_1^4 g(s)\,ds = 5, \quad g(1) = -1, \quad g'(1) = 2, \quad g(2) = -2, \quad g'(2) = \pi$$

you can still answer the questions below:

(a) Evaluate $I = \displaystyle\int_1^4 \frac{g(\sqrt{x})}{\sqrt{x}}\,dx =$

(b) Evaluate $I = \displaystyle\int_1^2 x g'(x)\,dx =$