Class 21: Wednesday October 22

Fundamental Theorem of Calculus

What is so *fundamental* about the Fundamental Theorem of Calculus? Generally speaking, it is that the FTC expresses an essential connection between two seemingly different types of inquiries: problems of rate of change and problems of accumulation. Here are three views of the FTC. You should be able to use the theorem in all three ways.

I. Rate of change of accumulation functions.

Theorem: Let $f(x)$ be a continuous function for $a \le x \le b$. Let $F(x)$ be the associated accumulation function defined by

$$F(x) = \int_a^x f(t)\,dt.$$

Then

$$F'(x) = f(x).$$

II. Solution of initial value problems.

Theorem: Let $f(x)$ be a continuous function for $a \le x \le b$. Define an initial value problem for $a \le x \le b$,

$$\begin{aligned} \frac{dy}{dx} &= f(x) \\ y(a) &= c. \end{aligned}$$

Then the solution to the initial value problem is given by

$$y = y(x) = \int_a^x f(t)\,dt \; + \; c.$$

III. Integral of a derivative.

Theorem: Let $F(x)$ be a differentiable function for $a \le x \le b$. Let $F'(x) = f(x)$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Using the Fundamental Theorem of Calculus
... to find derivatives of accumulation functions.

1. Is $\int_2^x \sqrt{1+\ln(t)}\,dt$ a function of x? If so, describe this function in words.

2. Find $\frac{d}{dx}\int_2^x \sqrt{1+\ln(t)}\,dt$.

3. Does $\frac{d}{dx}\int_2^x \sqrt{1+\ln(t)}\,dt = \frac{d}{dx}\int_1^x \sqrt{1+\ln(t)}\,dt$? Does this make sense?

4. Find the critical points of $F(x) = \int_0^x \sqrt{1+\cos(t)}\,dt$ on the interval $0 \le x \le 2\pi$.

Using the Fundamental Theorem of Calculus
... to find solutions to initial value problems.

1. Consider the initial value problem

$$y' = \sin(2x), \text{ where } y(\pi) = 1.$$

Which of the following functions solves this problem? (Consider all of them.)

a. $f(x) = \cos(2x)$

b. $g(x) = \int_{\pi}^{x} \sin(2t)\, dt \; + \; 2$

c. $h(x) = \frac{1}{2}\cos(2x) + 2$

2. Find the value of the constant c so that $f(x) = \sin(x) + c$ solves:

$$y' = \cos(x), \text{ where } y(\frac{\pi}{2}) = 0.$$

3. Write an accumulation function solving the initial value problem :

$$y' = \cos(x), \text{ where } y(\frac{\pi}{2}) = 0.$$

Using the Fundamental Theorem of Calculus
... to find evaluate definite integrals.

1. Which of these functions are antiderivatives of $f(x) = \cos^2 x = (\cos x)^2$?

   a. $F(x) = \dfrac{1}{2}x + \dfrac{1}{4}\sin 2x$

   b. $G(x) = \dfrac{1}{2}x + \dfrac{1}{4}\cos 2x$

   c. $H(x) = \dfrac{1}{2}x + \dfrac{1}{4}\sin x \cos x$

2. Use antiderivatives to evaluate $\displaystyle\int_{\pi/4}^{\pi/3} \cos^2 x \; dx$.