

Velocity and Distance
Suppose you are driving on back roads through the Rocky Mountains and neither the odometer nor the gas guage are working in your vehicle. Fortunately, the speedometer is working and you are driving with a friend who has a watch and who has some facility with paper and a pencil. Next we examine how a sequence of speedometer readings, say, one speedometer reading each minute, along with the relationship between distance and velocity, can be used to approximate the distance you have travelled.

1. If you drive with constant velocity \( v \) feet/sec for \( \Delta t \) seconds, how far have you gone?

2. Suppose \( v(t) \) is not constant, but that \( v(t) \) changes very little on the time interval \( T \leq t \leq T + \Delta t \). Approximately how far have you driven during this time interval?

3. If you drive a distance \( s_1 \) and then drive a distance \( s_2 \), how far have you driven altogether?

4. Use the ideas we have considered in this class so far, along with your answers to the questions above, to estimate the distance travelled by a car whose velocity satisfies the following table. Obtain both an underestimate and an overestimate of this distance.

| Time (sec) | 0 | 1 | 2 | 3 | 4 | 5 |
| Velocity (feet/sec) | 20 | 30 | 38 | 44 | 48 | 50 |

5. How could your estimates be improved?

Riemann Sums
Suppose a function \( f \) is defined on an interval \( a \leq t \leq b \). Suppose the interval is divided up into \( n \) subintervals of lengths \( \Delta t_1, \ldots, \Delta t_n \). Suppose \( t_k \) is a point chosen from the subinterval \( \Delta t_k \) for each \( k = 1, 2, \ldots, n \). A Riemann sum for \( f \) on the interval \([a, b]\) is a sum of the form

\[
\sum_{k=1}^{n} f(t_k)\Delta t_k = f(t_1)\Delta t_1 + f(t_2)\Delta t_2 + \cdots + f(t_n)\Delta t_n.
\]

If the left endpoint of each subinterval is chosen, the Riemann sum is a left-hand sum.

If the right endpoint of each subinterval is chosen, the Riemann sum is a right-hand sum.

Riemann Sums give us approximations to what property of the function \( f(x) \)?
Example 1: Velocity to Distance
Suppose $v(t)$ is a function that describes the velocity of a moving particle, in meters per second, at a time $t$ seconds after some starting time $t = 0$. What is the interpretation of the accumulation function $D(x)$ which is the accumulation of velocity with time?

Assume that the velocity of a moving particle is given by the following piecewise linear function:

$$v(t) = \begin{cases} 
10, & \text{for } 0 \leq t < 5 \\
2t, & \text{for } 5 \leq t < 40 \\
80, & \text{for } 40 \leq t < 50.
\end{cases}$$

Determine the appropriate formulas which define the accumulation function $D(x)$ over the three intervals:

$$D(x) = \begin{cases} 
& \text{for } 0 \leq x < 5 \\
& \text{for } 5 \leq x < 40 \\
& \text{for } 40 \leq x < 50.
\end{cases}$$

Example 2: Electrical Power and Energy
The work done by electrical power over some period of time, in illuminating light-bulbs for example, is referred to as electrical energy; energy = power $\times$ elapsed time. If power is measured in megawatts and time in hours, electrical energy is measured in megawatt-hours (MWh). See the graph below, from CiC p. 354.

Make a table with headings $T$ hours and $E(T)$ MWh. Here $E(T)$ is the total accumulated energy between 0 hours and $T$ hours. Using the graph of the power function above, evaluate $E(T)$ for $T = 0, 6, 9.5, 15, 21, \text{ and } 24$ hours. Then plot $E(T)$.

What is the relationship between $E(T)$ and $p(t)$? Is this similar or different from the relationship between $D(x)$ and $v(t)$?

We can say that and are examples of ACCUMULATION FUNCTIONS.
DEFINITIONS.
The definite integral. This evaluates to a number. For example,

$$\int_2^5 f(x)dx = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k)\Delta x, \quad 2 \leq x_1 < \ldots < x_n \leq 5, \quad \Delta x = (5 - 2)/n.$$  

Now realizing the specific upper limit of integration, 5, in the definite integral may be replaced by a variable gives us an accumulation function:

$$F(x) = \int_2^x f(t)dt = \lim_{n \to \infty} \sum_{k=1}^{n} f(t_k)\Delta t, \quad 2 \leq t_1 < \ldots < t_n \leq x, \quad \Delta t = (x - 2)/n.$$  

Note that the variable $x$ used for the interval endpoint is not the same as the “dummy variable” $t$ appearing in the integral. Replacing $x$ with a particular number specifies an interval of integration. Provided $f$ is (Riemann) integrable on this interval, the corresponding definite integral can be evaluated to produce a unique number. Each input to the accumulation function determines a unique output value. In this context, our geometric interpretation of the definite integral helps us to see that the formula

$$F(x) = \int_{a}^{x} f(t) dt,$$

defines a function with independent variable $x$, by giving us an intuitive understanding of the output of the function: if any specific value of $x$ is plugged in, say $x = 4$, then $F(4)$ is the signed area trapped between the curve $f(t)$ and the x-axis from $x = a$ to $x = 4$.

Write down an expression which relates the distance travelled $D(x)$ and the velocity $v(t)$

Write down an expression which relates the electrical energy used $E(T)$ and the power function $p(t)$

What are the differences between an accumulation function and a definite integral?