Class 9: Friday September 19

Local Linearity, Differentiability and the Microscope Approximation

Locally Linear Functions. A function is *locally linear at a point* $x = a$ if the function looks more and more linear on smaller and smaller intervals containing a.

Warm-Up Exercises: Where are the following functions locally linear?

$f(x) = e^{x} - 3$

$g(x) = \ln(x + 2)$

$h(x) = |5\sin(x)|$

$$k(x) = \begin{cases} 0 & \text{if } x \text{ is a rational number} \\ 1 & \text{if } x \text{ is an irrational number} \end{cases}$$

Remember: a number is *rational* if it can be written in the form a/b where a and b are integers — it is *irrational* otherwise. So π and e are irrational numbers. FACT: Between any two rational numbers is an irrational number. Between any two irrational numbers is a rational number.

Differentiability

The *derivative (slope, (instantaneous) rate of change)* of a function f at $x = a$ is defined as:

$$\begin{aligned} f'(a) &= \lim_{\Delta x \to 0} \frac{f(a) - f(a - \Delta x)}{\Delta x} && \text{Backward Difference} \\ &= \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x} && \text{Forward Difference} \\ &= \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a - \Delta x)}{2\Delta x} && \text{Centered Difference} \end{aligned}$$

provided that all of these limits exist and are equal. In this case, f is said to be *differentiable at* $x = a$.

The limit of the centered difference quotient as $\Delta x \to 0$ and the limit of the one-sided difference quotients as $\Delta x \to 0$ may not always exist and, if they exist, they may not always be the same.

Example. Sketch the following function and consider the point $(3, -6)$.

$$f(x) = \begin{cases} -2x & \text{if } x \le 3 \\ x - 9 & \text{if } x > 3 \end{cases}$$

a) At x=3, does the limit of central difference quotients as $\Delta x \to 0$ exist? What is that value?

b) At x=3, does the limit of left-sided difference quotients as $\Delta x \to 0$ exist? What is that value?

c) At x=3, does the limit of right-sided difference quotients as $\Delta x \to 0$ exist? What is that value?

d) Are all of the above limits the same? What do you conclude?

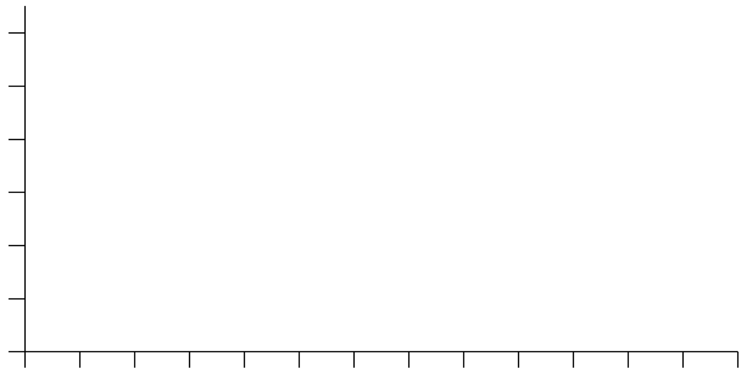
GROUPWORK: Discuss the following question with your nearest neighbor.

Is local linearity of a function at a point the same as a function being differentiable at a point?

Linear Approximations (Microscope Approximations)

If a function f is locally linear at a point a, we can approximate the function near a by the equation of the line through $(a, f(a))$ with slope $f'(a)$.
Essentially, we are approximating with the equation of the line that we see when we restrict our field of view with the microscope near a.

a) Make a rough sketch of the graph $f(x) = \sqrt{x}$.



b) Use the fact that $f'(4) = 1/4$ to find a linear approximation to $f(x)$ at the point (4,2).

(This linear approximation is sometimes called the *tangent line to the function at* $x = 4$.)
c) Estimate $\sqrt{4.0036}$ using the linear approxmation from part (b)

d) How close to the true value of $\sqrt{4.0036}$ is our estimate using the linear approximation?

Definition: The Microscope Equation relates a change in the input to a change in the output based on our linear approximation at a point:

$$\Delta f \approx f'(a) \cdot \Delta x$$