Locally Linear Functions. A function is locally linear at a point \( x = a \) if the function looks more and more linear on smaller and smaller intervals containing \( a \).

**Warm-Up Exercises**: Where are the following functions locally linear?

\[
f(x) = e^x
\]

\[
g(x) = \ln(x + 2)
\]

\[
h(x) = |5\sin(x)|
\]

\[
k(x) = \begin{cases} 
0 & \text{if } x \text{ is a rational number} \\
1 & \text{if } x \text{ is an irrational number}
\end{cases}
\]

Remember: a number is **rational** if it can be written in the form \( a/b \) where \( a \) and \( b \) are integers — it is **irrational** otherwise. So \( \pi \) and \( e \) are irrational numbers. FACT: Between any two rational numbers is an irrational number. Between any two irrational numbers is a rational number.

**Differentiability**

The derivative (slope, instantaneous rate of change) of a function \( f \) at \( x = a \) is defined as:

\[
f'(a) = \lim_{\Delta x \to 0} \frac{f(a) - f(a - \Delta x)}{\Delta x} \quad \text{Backward Difference}
\]

\[
= \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x} \quad \text{Forward Difference}
\]

\[
= \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a - \Delta x)}{2\Delta x} \quad \text{Centered Difference}
\]

provided that all of these limits exist and are equal. In this case, \( f \) is said to be **differentiable at** \( x = a \).

The limit of the centered difference quotient as \( \Delta x \to 0 \) and the limit of the one-sided difference quotients as \( \Delta x \to 0 \) may not always exist and, if they exist, they may not always be the same.

**Example.** Sketch the following function and consider the point \((3, -6)\).

\[
f(x) = \begin{cases} 
-2x & \text{if } x \leq 3 \\
x - 9 & \text{if } x > 3
\end{cases}
\]

a) At \( x = 3 \), does the limit of central difference quotients as \( \Delta x \to 0 \) exist? What is that value?

b) At \( x = 3 \), does the limit of left-sided difference quotients as \( \Delta x \to 0 \) exist? What is that value?

c) At \( x = 3 \), does the limit of right-sided difference quotients as \( \Delta x \to 0 \) exist? What is that value?

d) Are all of the above limits the same? What do you conclude?

**GROUP WORK**: Discuss the following question with your nearest neighbor.

Is local linearity of a function at a point the same as a function being differentiable at a point?
Linear Approximations (Microscope Approximations)

If a function $f$ is locally linear at a point $a$, we can approximate the function near $a$ by the equation of the line through $(a, f(a))$ with slope $f'(a)$.

Essentially, we are approximating with the equation of the line that we see when we restrict our field of view with the microscope near $a$.

a) Make a rough sketch of the graph $f(x) = \sqrt{x}$.

b) Use the fact that $f'(4) = 1/4$ to find a linear approximation to $f(x)$ at the point $(4, 2)$.

(This linear approximation is sometimes called the tangent line to the function at $x = 4$.)

c) Estimate $\sqrt{4.0036}$ using the linear approximation from part (b)

d) How close to the true value of $\sqrt{4.0036}$ is our estimate using the linear approximation?

Definition: The Microscope Equation relates a change in the input to a change in the output based on our linear approximation at a point:

$$\Delta f \approx f'(a) \cdot \Delta x$$