NAMES: ________________________________

Lab 8: Investigating Taylor Polynomials

Introduction

You should have read §10.2 before completing this lab.
Today we will use the computer algebra program `Derive`, found in the **Fowler Applications** folder to calculate Taylor polynomials for us. In particular we will explore regions of where a Taylor polynomial of a function fits the function well.

Work through the lab, making sure each member of your team understands what Derive is doing and how to manipulate Derive to solve the problems for you.

I. Calculating by hand.

In the space below, find the nth order Taylor polynomial that approximates the given function $f(x)$ about a given point $x = a$:

1. $f(x) = \cos x$ about $x = 0$, $n = 2$

2. $f(x) = \ln x$ about $x = 1$, $n = 4$

II. Calculating with Derive.

In the Mathematics folder on the computer, select `DERIVE`. You should see a blank page with the list of commands `Author, Simplify, Solve`, etc. in the top margin of the page.

To get the 2nd order Taylor polynomial that approximates $\cos x$ about $x = 0$:

- In the `Author` menu, select `Expression`, and enter the expression $cos(x)$. Once you've pressed `Ok`, you will see the expression on the page in line 1, highlighted in blue.
- In the `Calculus` menu, select `Taylor series`. Adjust the variable, expansion point, and order as needed.

  Once you've pressed `Ok`, you will obtain something like: TAYLOR(COS (x),x,0,2)
- To see the actual polynomial, under the `Simplify` menu, select `Basic`.

Check the polynomial you calculated by hand.

Let another lab partner go through the same process to check the polynomial for $\ln x$ about $x = 1$. (It will look a bit different in Derive.)

And, just for good measure, check your answers to exercises 1 and 6 on page 548. (Note that to integrate an expression, go to the `Calculus` menu and select `Integrate`.)

1. Find the seventh degree Taylor polynomial centered at $x = 0$ for the indicated antiderivatives

$$\int \frac{\sin x}{x}\,dx \qquad \int e^{x^2}\,dx \qquad \int \sin(x^2)\,dx$$

6. Find the seventh degree Taylor polynomial centered at $x = \pi$ for

$$\sin(x) \qquad \cos(x) \qquad \sin(3x)$$

(Once you have checked these polynomials, go to the `Edit` menu and use the `Remove` option to get rid of all the algebra lines for these calculations from the screen as we will first focus on $\cos x$.)

III. Graphing polynomials.

In the Algebra window, you should have a line with $\cos x$ defined and the Taylor polynomial of degree 2 written out. Let's plot these two functions together.

- Highlight the $\cos x$ with the mouse.
- Click on the graph icon at the top right of the screen (it resembles a small graph). This produces a plot window. Click on the `Plot!` option and you should see a graph of $\cos(x)$.
- Return to the Algebra window by selecting `1 Algebra ???.MTH` from the `Window` menu. Highlight the Taylor polynomial.
- Return to the Graph window by selecting `2 2D-plot` from the `Window` menu. Click on the `Plot!` option and the graph of the polynomial will be added to the plot of $\cos(x)$.

Let's see how the Taylor polynomials provide better approximations of $\cos x$ as we increase the degree of the polynomial. Obtain Taylor polynomials of degree $n = 4, n = 6, n = 8$, etc. and then plot them on the same graph.

As you increase the degree, you will need to adjust the x-axis range which can be done in `Set` menu using the `Range` option.

1. Why did I only ask you to look at the Taylor polynomials of even degree?

2. What degree polynomial do you need to "match" the graph of $\cos x$ for at least one full period?

3. What do you think will happen as you take higher and higher degree Taylor polynomials?

IV. Finite region of convergence.

In the Algebra window, go to the `Edit` menu and remove all expressions and in the Plot window, go to the `Edit` menu and delete all graphs. We will now look at the function $f(x) = \ln x$ based at $x = 1$. Author the function and obtain its Taylor polynomial of degree 1 at $x = 1$. Plot both of these. (You will need to adjust the plot ranges to get the best view of the whole graph.)

Now create the Taylor polynomials of degree 2, 3, 4, 5, 6, etc and plot them. Look carefully at what is happening as you take higher and higher degree Taylor polynomials.

To further examine this , you may want to clear the graph, plot $\ln x$ along with the Taylor polynomials of degree 10 and degree 21.

1. Why does the following statement seem reasonable: The Taylor polynomials for $f(x) = \ln x$ based at $x = 1$ only approximate $\ln x$ in the interval $(0, 2)$.

2. You may ask "Well, if a Taylor polynomial for $\ln x$ is only good for x-values between 0 and 2, it isn't much good, is it?" Au contraire. Recall your rules for natural logs and reduce $\ln(2592) = \ln((\sqrt{2})^{10}(\sqrt{3})^{8})$ to a calculation involving natural logs of numbers less than 2.

3. Now estimate $\ln(2592)$ by using the 6th degree Taylor polynomial for $\ln x$ and compare the estimate to the actual value of $\ln(2592)$.

   To substitute a numerical value for x in an expression in Derive,

   – Highlight the expression and under the `Edit` menu, choose the `Copy` option.
   – Under the `Declare` menu, choose `Function Definition`. Let the function `Name and Arguments` be $P(x)$. Then in the `Definition` line, paste in the copied expression.
   – Under the `Simplify` menu, select `Approximate...` and in the pop-up window, type in, for instance, 10P(sqrt(2)) and click on `Approximate`.

   How good is the estimate of $\ln(2592)$?

The natural log function is "bad", i.e., undefined, at $x = 0$, and thus we might think that that is the reason the Taylor polynomials seem to fit on a finite interval. So let's consider the function

$$f(x) = \frac{1}{1 + x^2}.$$

Is this function "bad" anywhere?

Find the Taylor polynomials of degree $n = 2, 10, 50$ for f about $x = 0$. Center the plot area at the origin and graph the function $f(x)$ and the polynomials.

4. Why does the following statement seem reasonable: The Taylor polynomials for $f(x) = 1/(1 + x^2)$ based at $x = 0$ only approximate $f(x)$ in the interval $(-1, 1)$.

5. Consider the terms in a large order Taylor polynomial for $f(x)$. For $|x| > 1$, what can you say about the size of the successive terms in the polynomial? Can you use this observation to explain why the Taylor polynomial is not a good approximation of f for x-values outside the interval $(-1, 1)$?

V. Binomial series.

Consider the function $g(x) = (1 + x)^m$.

1. By hand, calculate the fourth degree Taylor polynomial for this function g centered at $x = 0$.

2. Now think about the nth degree Taylor polynomial for g. Note that if m is an integer greater than or equal to zero, the polynomial has all terms equal to 0 after the first $m+1$ terms. Why?

For any other value of m, the polynomial terms do not die out. Let's set $m = \frac{1}{2}$. In Derive, plot $g(x)$ and some of its Taylor polynomials centered at $x = 0$.

3. From what you see in the graphs, determine the range of x-values for which the Taylor polynomials provide a good approximation of g.

4. Use a Taylor polynomial to estimate $\sqrt{1.25}$ with an error of less than .001. What degree polynomial did you need in order to do this?

EXTRA CREDIT: Complex numbers and Taylor series.

Complex numbers are numbers of the form $r + si$ where r and s are real numbers and i is a symbol defined by the property $i \cdot i = -1$. Thus $i^3 = i^2 i = -i$, $i^4 = i^2 i^2 = (-1)(-1) = 1$, etc. How do these complex numbers work in functions that we use in calculus?

a. Let's consider e^{si}. We know

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots$$

For this to hold true when x is an imaginary number, show that

$$e^{si} = \cos(s) + i \sin(s).$$

b. Now explain why at a recent mathematics conference there were bumper stickers for sale proclaiming: "Mathematicians: We're # $-e^{-\pi i}$!!"

Assignment.

Hand in ONE neat copy of this lab handout per group. The lab is due on **Thursday, November 20**.