NAMES: ______________________________

Lab 5: Simpson's Rule

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§0 Introduction The integral of a function on an interval is defined to be the limit of the Riemann sums for the function. In some cases, it is possible to use the Fundamental Theorem of Calculus to bypass the definition and use antiderivatives to evaluate integrals. In other cases, finding antiderivatives is difficult or even impossible, and numerical methods are used instead. This lab illustrates one of the most-often used methods for evaluating integrals. Calculators such as the TI-85 use a variant of the method known as "Simpson's Rule."

§1 Rectangular Approximations We have seen the following three methods of approximating an integral using a Riemann sum.

Left-Endpoint: $\int_a^b f(x)\,dx \approx L = \sum_{k=1}^N f(x_k)\Delta x,$ using $x_k = a + (k-1)\Delta x$

Midpoint: $\int_a^b f(x)\,dx \approx M = \sum_{k=1}^N f(x_k)\Delta x,$ using $x_k = a + (k-\frac{1}{2})\Delta x$

Right-Endpoint: $\int_a^b f(x)\,dx \approx R = \sum_{k=1}^N f(x_k)\Delta x,$ using $x_k = a + (k)\Delta x$

Each of them can be viewed as **rectangular** approximations on each subinterval.

The three sketches below show the graph of a positive function on an interval $x_L \le x \le x_R$ of width Δx. Sketch in the rectangles corresponding to the left, midpoint, and right estimates. Write out these estimates as a formula.

Trapezoid approximations

1. **The trapezoid approximation.** On the sketches below, fill in the rectangles corresponding to the left and right estimates. On the sketch labeled "trapezoid," sketch in the slant line joining the points at the ends of the interval.

left	trapezoid	right

a. Calculate the area of the trapezoid in this approximation.

b. Calculate the average of the left and right approximations. You should arrive at the same formula as above.

The **trapezoid** approximation to the integral is formed by averaging the left and right approximations:

$$\textbf{Trapezoid:} \qquad \int_a^b f(x)\,dx \approx T = \tfrac{1}{2}(L+R)$$

2. **The midpoint estimate *as* a trapezoid.** The two sketches below show a rectangle and a trapezoid.

   a. Calculate the areas of the two figures. You should end up with the same amount. Does your formula use the slope of the slant side of the trapezoid?

   b. The sketch below shows the graph of a function and an approximation to its integral using a trapezoid *tangent* to the curve at the midpoint. Why is this approximation is the same as the midpoint estimate?

Comparing the trapezoid and midpoint approximations. The sketch below shows the graph over a small interval. Added to the sketch are two lines: the line joining the endpoints of the graph on this interval and the line tangent to the graph at the midpoint.

c. For this example, is the trapezoid approximation an over- or under-estimate? Shade in the area corresponding to the amount of the error.

d. Is the midpoint approximation too much or too little? Shade in the area corresponding to the amount of the error. (Use a different shading method or a different color.)

e. Are the two lines in the diagram parallel?

f. Will the trapezoid method always be an overestimate? . . .always an underestimate? What determines the "sense" of the approximation?

g. In this example, which is larger: the amount of the trapezoid error or the amount of the midpoint error?

§3 Numerical comparisons

3. A polynomial example.

   a. Use the Fundamental Theorem of Calculus to give the exact value of the following integral:

   $$\int_0^2 x^5 + 1 \, dx =$$

   b. Use the program `AGGSUM` to fill in the third line of the following table of estimates to the integral you just computed. Keep at least five places beyond the decimal point.

N	L	R	T	M
5				
20				
50				
100				

   Which of the methods is most accurate for this integral?

§4 Comparing errors numerically

The program `SUMCOMP` makes it easier to compare the errors for each of the four methods. Examine the listing of this program before running it. (Really!) You need to specify the formula for the function, the interval, and *the exact value of the integral.* The program prompts you for the number of subintervals and then reports the computed approximations and the "error" for each method.

The errors are computed as follows, using I as the exact value of the integral:

$$\begin{aligned} I - L &= \text{error for the left-endpoint method} \\ I - R &= \text{error for the right-endpoint method} \\ I - T &= \text{error for the trapezoid method} \\ I - M &= \text{error for the midpoint method} \end{aligned}$$

4. To verify the program is working correctly, use it to approximate the same integral as before. Choose N = 50 subdivisions. (Note that this program prompts you for N when you run it.) How do your results compare? Now fill in the rest of the table above.

5. Some of the errors are positive, some negative. Which sign corresponds to an overestimate, which to an underestimate?

Combining trapezoid and midpoint methods

6. By studying the errors in approximation in the trapezoid and midpoint methods, we can develop an even better approximation method.

   a. Use `SUMCOMP` to complete the following table of errors.

   $I = \int_0^2 x^2\, dx =$ ____________________ (Fill in the exact value.)

N	$I - T$	$I - M$
5		
20		
50		

   b. Now calculate a table of errors for two integrals of your choice: one a polynomial of degree greater than 2 and one non-polynomial.

   $I = \int \qquad\qquad dx =$ ____________________

N	$I - T$	$I - M$
5		
20		
50		

   $I = \int \qquad\qquad dx =$ ____________________

N	$I - T$	$I - M$
5		
20		
50		

7. Now study the patterns in the errors you have just calculated.

   a. How do the *signs* of the errors compare?

   b. How do the *magnitudes* of the errors compare?

   c. Calculate the ratio $\dfrac{I-T}{I-M}$ for some of the examples above. What pattern do you see?

8. Use the previous result to explain why $\frac{2}{3}M + \frac{1}{3}T$ should be a better approximation to I than either M or T.

§5 Simpson's Rule

The estimate $\frac{2}{3}M + \frac{1}{3}T$ is called "Simpson's Rule."

$$\textbf{Simpson:} \qquad \int_a^b f(x)\, dx \approx S = \frac{2}{3}M + \frac{1}{3}T$$

The program `SIMPSON` uses this method to calculate an approximation of an integral. Use this program to calculate successive approximations to at least some of the following integrals (all, if you can). Successively choose values of N until you have stable values to three decimal places. (We have not given an analysis of when the Simpson approximation is an over- or under-estimate.)

1. $I = \int_0^2 -4x + 1\, dx =$ ____________

N	Simpson Approximation

2. $I = \int_0^2 3x^2 - x + 1\, dx =$ ____________

N	Simpson Approximation

3. $I = \int_0^2 4x^3 - 6x + 1\, dx =$ ____________

N	Simpson Approximation

4. $I = \int_0^2 x^3 + 2x^2\, dx =$ ____________

N	Simpson Approximation

5. $I = \int_0^3 x^4 - 2x^2 \, dx =$ ____________

N	Simpson Approximation

6. $I = \int_0^2 \ln(x^2 + 1) \, dx =$ ____________

N	Simpson Approximation

7. $I = \int_0^2 \sqrt{4 - x^2} \, dx =$ ____________

N	Simpson Approximation

8. $I = \int_0^\pi 2^{\sin(x)} \, dx =$ ____________

N	Simpson Approximation

9. $I = \int_0^\pi \cos^2 x \, dx =$ ____________

N	Simpson Approximation

10. $I = \int_3^5 x^x \, dx =$ ____________

N	Simpson Approximation

§6 Assignment

Lab Essay: Each team must turn in one **neat** copy of this worksheet, completely filled out on **Thursday October 30**. In addition you should answer the following questions (probably on an attached sheet).

1. Why are Trapezoid Rule and the Midpoint Riemann sums exact when the integrand is linear and why is Simpson's Rule exact when the integrand is a polynomial of degree 3 or less?

2. How does the concavity of the integrand function, i.e. $f(x)$ in $\int_a^b f(x)dx$ determine whether the Trapezoid Rule and Midpoint Riemann sums will produce over- or under-estimates? How does does the slope of the integrand function determine the error made by Left Hand and Right Hand Riemann sums?

3. How does the error in the approximations made by the Trapezoid Rule, Midpoint and by Simpson's Rule depends on the size of N (correspondingly, the size of Δx)? [HINT: You might try graphing $\log(|error|)$ versus $\log(N)$.]