NAMES: ______________________________

Lab 2: Newton's Method

The purpose of this lab is to explore Newton's method for finding roots of functions. In particular we will develop a recursive formula for the method and determine some of the shortcomings of this method.

Recursive formula

Consider a general function $h(t)$. We would like to get an approximation of a root, t^* of this function, and although we do not know the exact value of t^*, we can give a rough first approximation of $t = t_0$.
Consider the line tangent to h at $t = t_0$.

1. What is the slope of this line? (While we cannot write it down numerically, we do have notation that describes this slope.)

2. Name a point that this tangent line must pass through. (Give both coordinates.)

3. Using the slope and point of the tangent line, determine the equation of the tangent line (in the slope-intercept form $y = mt + b$).

$$y = \underline{\hspace{6em}}\; t + [\underline{\hspace{12em}}].$$

4. Now find the root $t = t_1$ of this tangent line, i.e., where the line crosses the t-axis. Simplify the expression for t_1 as much as possible.

5. Repeat the process, finding an expression for the root $t = t_2$ of the equation of the line tangent to h at the point $t = t_1$.

6. Write a general recursive formula for the root $t = t_n$ of the equation of the line tangent to h at the point $t = t_{n-1}$. Check this formula with other lab teams and/or with the instructor.

Testing our formula

We will use the recursive formula on a specific case. Let $h(t) = t^3 - 2$.

1. Find $h'(t)$.

2. Start with $t_0 = 1.5$. Calculate $h(t_0)$ and $h'(t_0)$. Use the recursive formula to find t_1, the root of the line tangent to the graph of h at t_0.

3. Calculate $h(t_1)$ and compare the value with $h(t_0)$. Is t_0 or t_1 closer to the root of h? How do you know?

4. On a separate sheet, draw the graph of h illustrating the determination of t_1.

5. Repeat the Newton process and fill in the table below:

Step	t_n	$h(t_n)$	$h'(t_n)$
0	1.5		
1			
2			

Computer Iterations

Now that we have the hang of Newton's method by hand, we will get the computer to do the iterative process for us. Open the Excel spreadsheet `newton.xls` (in the `S:\Math Courses\Math118` directory) where the basic format is set up for you. Carefully fill in the formulae for the table, and then, as a check, compare your results above with those in the first few lines of spreadsheet.

1. Suppose we want to find the solutions of the equation $x^3 + 2x^2 - 10x = 20$. Define a function $f(x)$ such that the solutions of $x^3 + 2x^2 - 10x = 20$ are identical to the solutions of $f(x) = 0$, i.e., the roots of f. Once you have the definition of f, adjust the spreadsheet as needed and find all the roots. How are you sure that you have all the roots?

2. Now investigate the problem $x^3 + 3x^2 - 2x - 4 = 0$. Suggestion: start with $x_0 = 0$. What happens? Why? Has Newton's method failed us, or were we just unlucky to pick this starting point? Draw a sketch of the graph of your function and illustrate what is going on with the recursive process.

3. Now consider the equation $\cos(x) = x$. Define f appropriately in the spreadsheet and start with $x_0 = 16$. What is happening? Again, draw a sketch of the graph of your function and illustrate what is going on.

Algebraic Exploration

1. Consider the equation $x^2 - A = 0$. Show that Newton's method is the same as the Babylonian method discussed in class.

2. Consider the equation $x^m - A = 0$. Show that Newton's method can be used to produce a Generalized Babylonian algorithm which produces an approximation to $\sqrt[m]{A} = A^{1/m}$. The Generalized Baylonian Algorithm is $x_{n+1} = \dfrac{1}{m}\left((m-1)x_n + \dfrac{A}{x_n^{m-1}}\right)$. (Check that $m = 2$ corresponds to your answer in **1.**) Use the Generalized Babylonian Algorithm to estimate $\sqrt[5]{7}$ to 9 decimal places.

Assignment

Each team will turn in one neat copy of this lab with the signature of all the group members on the coversheet. This is due in lab on **Thursday September 25**.