**NAMES:** ______________________________

# Lab 1: Euler's Method and the SIR Model

## Introduction

An Initial Value Problem often contains information about its solutions which one can find without necessarily knowing what the solutions are explicitly. Sometimes a qualitative analysis tells us something about the dynamics of the situation the IVP models.

## §1. The SIR model and the peak of an epidemic

For this activity, we will consider the SIR model

$$\begin{aligned} S' &= -aSI \\ I' &= aSI - bI \\ R' &= bI \\ S(0) &= S_0 \\ I(0) &= I_0 \\ R(0) &= R_0, \end{aligned}$$

with the specific initial conditions provided in the Excel file `SIRmodel.xls`. The file can be found in the `S:\Math Courses\Math118` directory. You will find two sheets in this file, one labeled `SIR Model and Euler's Table` and the other labeled `S, I and R versus time`.

When an illness passes through a population, it is said to be an *epidemic* as long as the size of the infected population is increasing, that is to say, when more people are getting sick each day than are recovering. Once the number of recoveries each day exceeds the number of new cases of the illness, the disease is no longer considered an epidemic. The illness has *peaked* when the number of infected persons is greatest. Before its peak, a disease is considered an epidemic; after the peak, it is no longer thought of as an epidemic. It is possible for a disease to avoid detection until after its peak has passed.

1. For the given SIR model, create the Euler's Method table on the sheet labeled `SIR IVP and Euler's Table`. Make the table large enough so that it covers the first 100 days of the epidemic. The value for $\Delta t$ is given as 1. Once you have produced the Euler's table, check out the graph on the sheet `Time versus S, I and R`.

2. By hovering the mouse arrow over the desired point on the graph, find the values of $S, I, R$ and $t$ when the disease has peaked. Enter these numbers in the first row of the table on the next page.

| Disease Peak | | | | | | | |
|---|---|---|---|---|---|---|---|
| Initial Conditions | | | Data at Peak | | | | |
| $S(0)$ | $I(0)$ | $R(0)$ | $t$ | $S$ | $I$ | $R$ | no peak |
| 7500 | 200 | 30 | | | | | |
| 5000 | 200 | 200 | | | | | |
| 5000 | 5000 | 30 | | | | | |
| 1000 | 200 | 2000 | | | | | |
| 2000 | 30 | 7000 | | | | | |
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3. Whether a disease has an epidemic stage or not depends on a number of things, including how infectious the disease is and how quickly an infected person recovers. But it may appear to depend on the initial populations of susceptible, infected and recovered individuals.

   In your groups, investigate the relationship between the initial conditions and the occurrence of an epidemic. Each of the first five rows in the table above has different initial conditions. Make these changes in the spreadsheet and find the data at the peak for each set of conditions. Once you have finished, examine your data and make a conjecture regarding the occurrence of an epidemic.

   Test your conjecture by making up different initial conditions, modifying the spreadsheet and using the graph to record the peak data.

   Reexamine your conjecture. Was it justified? If not, how would you now modify it?

## §2. The threshold value of an epidemic

Here, we study the effect of the initial conditions on the occurrence of an epidemic using an analytic or symbolic approach. This means we focus on the differential equations that define the epidemic.

1. Suppose the population of infected individuals has a peak. At the time of the peak, what can be said about $S'$, $I'$ or $R'$? Use this observation of the value of one of these derivatives to determine what you can about the size of $S$, $I$, and/or $R$ at the time of the peak of the epidemic.

2. The $S$ value at the peak of the epidemic is called the *threshold* value. Suppose the initial susceptible population is below the threshold value. What happens to the epidemic?

3. Does the computed threshold value agree with your numerical approximations? Why or why not? How could you modify the Euler's table to improve your approximations?

4. Now we will adjust the value of $\Delta t$ to .5, .25, .1, .05, etc. As you decrease the value of $\Delta t$, consider the following questions: How do the values of $I$ and $t$ at the peak compare? What would explain your observations?

## §3. Immunity loss in the $SIR$ model

In class, we have yet considered the phenomenon of immunity loss and how that would change the $SIR$ model (see CIC 22-23 in the electronic reserves). Using the values $a = 0.00004, b = 1/5, c = 1/20$ for the coefficients, modify the equations in the Excel to include immunity loss. (Adjust $\Delta t$ so that you have a nice SMOOTH graph.) Use the initial conditions $S_0 = 7500, I_0 = 500, R_0 = 2000$ for this section.

1. What is different about the graphs of $S$, $I$ and $R$ with immunity loss as compared to the original model? How do the graphs reflect the situation with immunity loss?

2. What is the effect of immunity loss on the population in the long run? How do you know?

**EXTRA CREDIT**
Try exploring your immunity loss model using Euler's Method with a very small $\Delta t$ and a very large $N$ to simulate the long term behavior of the S-I-R disease on the population. Can you come up with equations which predict what the eventual populations of the susceptibles, infecteds and recovereds will be when immunity loss is included, i.e. determine $S_\infty$, $I_\infty$ and $R_\infty$. HINT: Do you see any relationships between these values and their ratios (i.e. $R_\infty/I_\infty$ etc) and the model's parameters and their ratios (i.e. $b/a$ and $b/c$)?

## §4. Assignment

Each team will turn in one collaborative lab report that reflects the *entire* team's understanding of the lab. The report is due in two weeks,**Thursday, September 25 in lab**. (Usually you will only have one week to turn in a lab report.)
Explanations should be clear, complete, phrased in your own words, and written in good English. There should be only *one* topic per paragraph. Feel free to include any data, tables, or graphs that you need to help with your explanations, but be sure to label them and refer to them in the text of your report (otherwise I won't know to look at them). Your report should be *typed* and no more than four pages in length. Any math symbols, graphs, or tables can be written in neatly by hand. Please refer to the *Comments on Lab Team Writing Assignments* handout for more details and guidance.
Your report has three parts (corresponding to the sets of questions written below), and should be labeled I, II and III. Do not consider these questions an outline of your report; they are open-ended to get you to ponder all sides of the problem and all approaches to the problem (numerical, graphical and symbolic). You can assign parts of the report to individual members of the team but recall that the entire report is a group project and all parts should reflect the input and oversight of the whole group.

I. In the course of this lab you have made some discoveries about the conditions necessary for a disease modelled by $SIR$ to be an epidemic. Detail your discovery and provide your reasoning and evidence to support your conclusions regarding the threshold value and when, how and if a $SIR$-modelled disease will be epidemic.

II. In studying the $SIR$ model, we have used Euler's method. How does Euler's method work? (This is the meat of the report, and it takes careful work to put Euler's method into words accurately.) What changes occur in the results of Euler's method from changes in $\Delta t$? Why do these changes occur? At what point do you feel that changing $\Delta t$ is no longer helpful? Why?

III. In the model with immunity loss, what did you see happening? (Note that this requires you to organize a lot of observations in your report.) Why? (Can you support your numerical and graphic observations from the computer simulations with symbolic manipulations of the model?)