*Class 32*: Friday November 22

# Convergence Tests for Infinite Series

## Geometric Series

We know that the geometric series $\sum_{k=0}^{\infty} ar^k$ converges when $|r| < 1$ and we know what it converges to (which is extremely rare): $\dfrac{a}{1-r}$.

This is very interesting, because it shows that you can add up an infinite list of numbers, and obtain a finite answer. In the geometric series example it is relatively easy to find the condition which tells us when the sum converges to a finite answer. The question we are interested in, is given an infinite series, how can we prove it will converge or diverge?

GROUPWORK

Consider the list of numbers below....

$$1, \qquad 1/2^2, \qquad 1/3^3, \qquad 1/4^4, \qquad 1/5^5, \qquad 1/6^6, \qquad \textit{etc.}$$

In small groups use your calculators to begin with the first number on the infinite list above, 1, and progressively add each successive number on the list, keeping track of the subtotals you get by placing them in the chart below, with seven places after the decimal.

| $n$ | $n^{\text{th}}$ | subtotal |
|---|---|---|
| 1 | 1 | $= 1.0000000\ldots$ |
| 2 | $1 + 1/2^2$ | $=$ |
| 3 | $3^{\text{rd}}$ subtotal | $=$ |
| 4 | $4^{\text{th}}$ subtotal | $=$ |
| 5 | $5^{\text{th}}$ subtotal | $=$ |
| 6 | $6^{\text{th}}$ subtotal | $=$ |
| 7 | $7^{\text{th}}$ subtotal | $=$ |
| $\vdots$ | $\vdots$ | |
| $\infty$ | Final Sum | $=$ |

What do you find happening to the subtotals? If this trend continues, what will be the first four digits of all the subtotals beyond those in the table? None of the numbers in the list, beyond a certain point, seem to be affecting the first four digits of the subtotals. So, if you were somehow able to add up *all* of the numbers in the infinite list, what do you think the first four digits of the total would be?

Find the first six decimals of the sum of the numbers in our infinite list.

What would you do to find the first ten decimals of the sum of the numbers in ou r infinite list? (You don't have to actually do it.)

How would you describe the sum of our infinite list of numbers using the concept of "limit"?

**Formal Language of Infinite Series.**
Using the proper terminology, we will discuss what you have just done. We had a list of numbers (which is called a *sequence* of numbers):

$$1, \qquad 1/2^2, \qquad 1/3^3, \qquad 1/4^4, \qquad 1/5^5, \qquad 1/6^6, \qquad \text{etc.}$$

which we call the **TERMS** of the **INFINITE SERIES**

$$1 + 1/2^2 + 1/3^3 + 1/4^4 + 1/5^5 + 1/6^6 + \ldots = \sum_{k=1}^{\infty} 1/k^k.$$

We tried to find the sum of this infinite series by looking at its **SEQUENCE OF PARTIAL SUMS** (list of subtotals):

$$\begin{aligned}
S_1 &= 1 \\
S_2 &= 1 + 1/2^2 \\
S_3 &= 1 + 1/2^2 + 1/3^3 \\
S_4 &= 1 + 1/2^2 + 1/3^3 + 1/4^4 \\
&\vdots \\
S_n &= 1 + 1/2^2 + 1/3^3 + 1/4^4 + \ldots + 1/n^n \\
&\vdots
\end{aligned}$$

We found that the sequence of partial sums $S_n$ seemed to have a **LIMIT** (the subtotals were stabilizing), and that the limit of this sequence of partial sums was the **SUM** of the infinite series:

$$\sum_{k=1}^{\infty} 1/k^k = \lim_{n \to \infty} S_n.$$

When the partial sums $S_n$ of an infinite series have a limit, the infinite ser ies is said to **CONVERGE**.
When the partial sums $S_n$ do not have a limit, the infinite series is said to **DIVERGE**.
Therefore, in this case, the infinite series that we have been examining converges.

GROUPWORK
In small groups, try and complete the following sentences.

## General Principles for Convergence of Series.

- If the individual terms of an infinite series do not approach 0, then the infinite series will

- If an infinite series converges, then the individual terms of the infinite series must

- If the terms of an infinite series approach 0, will the infinite series necessarily converge?

**Example 2** $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ (This is called the **HARMONIC SERIES**.)

Partial sums (fill in the sums):

$S_1 = 1 =$

$S_2 = 1 + 1/2 =$

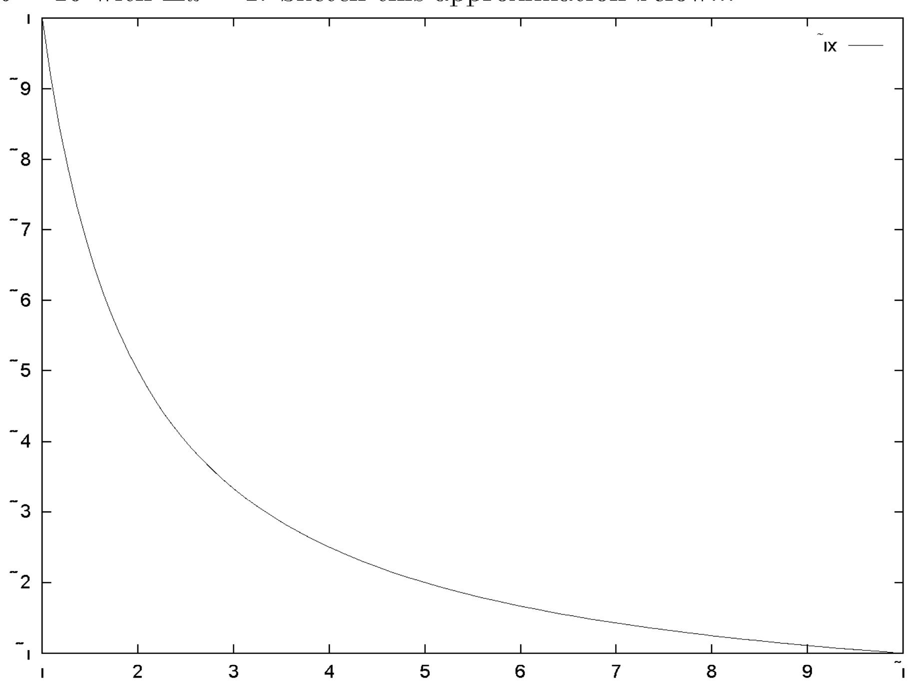
$S_3 = 1 + 1/2 + 1/3 =$

$S_4 = 1 + 1/2 + 1/3 + 1/4 =$

Do you think these partial sums have a limit?

We need to come up with a systematic way of determining the convergence or divergence of an infinite series. Over the next few days we will learn about **Convergence Tests**.

Let us look at the Left-hand Riemann Sum approximation **L** of the area under the curve $f(x) = 1/x$ from $a = 1$ up to $b = 10$ with $\Delta x = 1$. Sketch this approximation below...




Is **L** an over-estimate or an under-estimate?

What is the relationship between the Left-hand Riemann Sum, $S_{10}$ and the $\int_1^{10} \frac{1}{x}\,dx$? Write in those relationships (<, > =, etc) below...

$L$ $\qquad\qquad$ $S_{10}$ $\qquad\qquad$ $\displaystyle\int_1^{10} \frac{1}{x}\,dx$

What happens if instead of 10 we sum up to 1000? 100000? Infinity?

So, by geometry we can show that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$, the HARMONIC SERIES, ____________________ .

**1. INTEGRAL TEST**

If $\int_1^\infty a(k)\,dk$ CONVERGES, then $\sum_{k=1}^\infty a(k)$ CONVERGES.

If $\int_1^\infty a(k)\,dk$ DIVERGES, then $\sum_{k=1}^\infty a(k)$ DIVERGES.

<u>GROUPWORK</u>

Determine whether the following infinite series CONVERGE or DIVERGE.

**<u>Example 3</u>** $\displaystyle\sum_{k=1}^\infty \frac{1}{k^2}$

**<u>Example 4</u>** $\displaystyle\sum_{k=1}^\infty k^2$

**<u>Example 5</u>** $\displaystyle\sum_{k=1}^\infty \frac{1}{\sqrt{k}}$

**Connection Between Improper Integrals of the First Kind and Infinite Series**

By applying the integral test to the infinite series $\displaystyle\sum_{k=1}^\infty \frac{1}{k^p}$ and reviewing the examples above fill in the appropriate condition on $p$ in the RULE below

$$\sum_{k=1}^\infty \frac{1}{k^p} \begin{cases} \textbf{CONVERGES} & \text{when } p \\ \textbf{DIVERGES} & \text{when } p \end{cases}$$