Class 31: Wednesday November 20

Infinite Series from Taylor Series

Consider the infinite series defined by

$$G(x) = 1 + x + x^2 + x^3 + x^4 + \ldots + x^n + \ldots$$

For which values of x does this series converge to a fixed finite value?

In trying to address this problem, the easiest first step is to look at the size of the individual terms of the series. If the terms are **not** heading toward 0, then the infinite sum of these terms is not fixed and finite, i.e., the series does not converge.

a. Suppose $x > 1$. What can you say about the terms of the series $G(x)$ for $x > 1$? What can you say about the convergence of the infinite series?

b. Suppose $x < -1$. What can you say about the terms of the series $G(x)$ for $x < -1$? What can you say about the convergence of the infinite series?

c. Suppose $x = 1$. Write out the first five terms of $G(1)$. What do you think about the convergence of the infinite series? And what can you say about the terms of the series $G(1)$? Do they get closer and closer to 0?

d. Suppose $x = -1$. Write out the first five terms of $G(-1)$. What do you think about the convergence of the infinite series? And what can you say about the terms of the series $G(-1)$? Do they get closer and closer to 0?

e. Suppose $-1 < x < 1$. What can you say about the terms of the series $G(x)$ for $x \in (-1, 1)$? What can you say about the convergence of the infinite series? (Be careful here!)

If our only observation of $G(x)$ is that the terms in the series approach 0 if $x \in (-1, 1)$, we do **not** have enough information to claim that the series converges. But at least we have some hope that $G(x)$ may be a finite, fixed value for $x \in (-1, 1)$.

Consider the partial sum of the infinite series:

$$S_n = 1 + x + x^2 + x^3 + x^4 + \ldots + x^n = \sum_{k=0}^{n} x^k.$$

Note that this sum S_n has only a finite number of terms and that S_n has the following relationship to our function $G(x)$:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=0}^{n} x^k = \sum_{k=0}^{\infty} x^k = G(x).$$

So we are going to try to figure out what S_n is, and then take the limit as $n \to \infty$. Fill in the blank:

$$S_n \quad = \quad 1 \quad + \quad x \quad + \quad x^2 \quad + \quad x^3 \quad + \quad \ldots \quad + \quad x^n$$

$$x \cdot S_n \quad = \quad \rule{10cm}{0.4pt}$$

Now in the space above, subtract the bottom equation from the top equation and simplify. Solve for S_n. (Note that this requires that $x \neq 1$. Why? Since we are interested in $-1 < x < 1$, we know that $x \neq 1$.)

$$S_n = \rule{4cm}{0.4pt}$$

Now, we take the limit of S_n as $n \to \infty$ to see what $G(x)$ looks like. Do not forget that $-1 < x < 1$.

$$G(x) = \lim_{n\to\infty} S_n = \rule{6cm}{0.4pt}$$

So here's the finale:

$$G(x) = \sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \qquad \text{for } x \in (-1, 1).$$

This answers even more than we asked back on page 1. There, we asked "For which values of x does this series converge to a fixed finite value?" We know now that the series converges only for $x \in (-1, 1)$.

However, we also know **exactly** what the value of the infinite series is! This is rare. In studying series, there are two big questions:

Does the series converge?

If so, what does it converge to?

The first question is much easier to answer than the second, but in this case of $G(x)$, we actually answered both!

Calculate the following infinite sums:

$1 + (-0.4) + (-0.4)^2 + (-0.4)^3 + \dots$

$\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots$

As you may have suspected, $G(x)$ is a special series. It is called the **Geometric series**, and it pops up frequently in mathematics.

Since our focus lately has been on Taylor series, we'll take a moment here to show the connection between the Geometric Series and a Taylor series.

From Lab 7, we saw the Binomial Series, which we constructed as a Taylor series of the function $(1+t)^m$ centered at $t = 0$:

$$(1+t)^m = 1 + mt + \frac{m(m-1)}{2}t^2 + \frac{m(m-1)(m-2)}{3!}t^3 + \dots$$

Let $m = -1$ and let $t = -x$. Then

$$(1+t)^m = \frac{1}{1-x}$$

Do the same substitutions in the series - what does the series look like?

So, the Geometric Series is the Taylor series centered at $x = 0$ for the function

$$G(x) = \frac{1}{1-x} \text{ for } -1 < x < 1.$$

(If you want some practice at constructing Taylor series, show directly that $1 + x + x^2 + \ldots$ is the Taylor series centered at $x = 0$ for this function.)

Let's summarize our study of convergence of the series for various x values. Draw a number line below, and indicate on it the x values for which the Taylor series converges and the x values for which the Taylor series does not converge.

You should see an interval of x values for which the Taylor series converges. This is called the **interval of convergence**. What is the center of this interval? How is it related to the center of the Taylor series?

Now consider the Taylor series

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \ldots$$

What is the condition for this series to converge? Is this a geometric series?

On what interval did the Taylor Series $P_n(x;0)$ approximate the function $f(x) = \dfrac{1}{1+x^2}$ well in Lab 7?

How are the answers to the previous two questions related?