

Errors in Taylor Polynomial Approximations

Consider the fourth order Taylor polynomial approximation of a function f for x -values near 0:

$$P_4(x) = \sum_{k=0}^4 \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \frac{f^{(4)}(0)}{24}x^4.$$

This is the best fit of f near $x = 0$ that we can get from a fourth degree polynomial. The goal for today is to determine how good the fit is. How well does $P_4(x)$ approximate $f(x)$ near $x = 0$?

We begin with an example. Consider the fourth order Taylor polynomial approximating the function $g(x) = e^x$ for x -values near 0. In the last class, we found

$$P_4(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4.$$

1. Define $y = g(x) - P_4(x)$. Thus y tells us the error of P_4 in approximating g . We will refer to y as the error function.
 - a. Using your calculator, graph y vs x in the plot box $-5 < x < 5$, $-15 < y < 15$. Sketch what you see. Describe the difference between $g(x)$ and $P_4(x)$ for $-2 < x < 2$.
 - b. Now zoom in on your graph, setting your x -range to $-2 < x < 2$. Before you graph this, be sure to adjust your y -range appropriately. Describe the difference between $g(x)$ and $P_4(x)$ for $-1 < x < 1$.
 - c. Zoom in again on your graph, setting your plot box to $-1 < x < 1$ and $-0.01 < y < 0.01$. Describe the difference between $g(x)$ and $P_4(x)$ for $-1 < x < 1$.

2. Now let's consider the shape of the graph of the error function y that you have been plotting.

Notice how the graph of the error function y looks like the graph of a cubic function. We'll assume that y is cubic and try to build a function to model y .

- a. Our first order of business is to assemble what we have observed about the values of the error function y near $x = 0$.

* When $x = 0$, we see from the definition of our error function

$$y = g(0) - P_4(0) = \underline{\hspace{2cm}}$$

* When $x = 1$, we see from the graph that the error $y \approx 0.01$. Verify this on the graph of y using the trace feature on your calculator.

- b. Then a cubic function that might model our error function y is $0.01x^3$. Does this satisfies the two observations we made above?

- c. Now graph this possible model function $y_2 = 0.01x^3$ along with the error function y on the plot box $-1 < x < 1$ and $-0.01 < y < 0.01$. How well does our model function represent the error function?

3. We see that the cubic function increased faster on the interval $0 < x < 1$ than our error function did. So we look for another odd power function (why odd?) that bows out more on this interval.

- a. Let's just try the function $0.01x^5$. Define this function on your calculator. Graph this with the error function y *and* the cubic model function on the same plot box and compare the graphs.

- b. Maybe the function $0.01x^7$ is even better. Graph this too and compare it with the other graphs.

4. a. These higher degree functions look promising as matches for the error function y , at least graphically from what can be seen on the little calculator screen. How about numerically? We will look at the ratio of error function and each of the three odd power functions. This will give us a comparison of the relative size of the two functions in the ratio. In your teams, divide up the work and fill in the following table with three decimal places of accuracy. (If you don't know how to use the functions on your calculator efficiently, see the note below.)

x	$\frac{e^x - P_4(x)}{0.01x^3}$	$\frac{e^x - P_4(x)}{0.01x^5}$	$\frac{e^x - P_4(x)}{0.01x^7}$
1.00			
0.50			
0.10			
0.05			
0.01			

Note: On your TI-83 calculator, you should have

$$\begin{aligned}
 Y1 &= e^x - (1 + x + x^2/2 + x^3/6 + x^4/24) \\
 Y2 &= 0.01x^3 \\
 Y3 &= 0.01x^5 \\
 Y4 &= 0.01x^7
 \end{aligned}$$

Use these function definitions to your advantage. For example, on the TI-83, if you are trying to find the value of the ratio $(e^x - P_4(x))/(0.01x^3)$ when $x = 0.5$, you can do the following steps:

- * Store 0.5 as the value of X : $(0.5 \text{ [STO} \rightarrow \text{] [X,T,} \Theta, n \text{] [ENTER]})$.
- * Press **[VARS]**, select Y-VARS and under the FUNCTION menu, select Y1. Then you should see Y1 on the home screen. We want to divide this by Y2, so press **[\div] [VARS]**, select Y-VARS and under the FUNCTION menu, select Y2. Now you should have Y1/Y2 on the home screen.
- * Press **[ENTER]**, and the calculator gives you the value of Y1/Y2 at $X=0.5$.

- b. Notice that, as $x \rightarrow 0$, each of the functions

$$e^x - P_4(x), \quad 0.01x^3, \quad 0.01x^5, \quad \text{and} \quad 0.01x^7$$

approaches 0. (We see this graphically as all our graphs pass through the origin.)

Now look at your table and answer the following using your numerical evidence.

- * As $x \rightarrow 0$,

$$\frac{e^x - P_4(x)}{0.01x^3} \rightarrow \underline{\hspace{2cm}}$$

So, as $x \rightarrow 0$, which function approaches 0 faster,

$$e^x - P_4(x) \quad \text{or} \quad 0.01x^3? \quad (\text{circle one})$$

- * As $x \rightarrow 0$,

$$\frac{e^x - P_4(x)}{0.01x^7} \rightarrow \underline{\hspace{2cm}}$$

So, as $x \rightarrow 0$, which function approaches 0 faster,

$$e^x - P_4(x) \quad \text{or} \quad 0.01x^7? \quad (\text{circle one})$$

- * As $x \rightarrow 0$,

$$\frac{e^x - P_4(x)}{0.01x^5} \rightarrow \underline{\hspace{2cm}}$$

Now this is an interesting case, because the ratio of the error function and the fifth degree model is not zooming to 0 or ∞ . In fact, it appears that the ratio is fairly constant. This implies that the error function and x^5 are strongly related as $x \rightarrow 0$.

- c. We say that the error function $e^x - P_4(x)$ has the same *order of magnitude* as x^5 as $x \rightarrow 0$ because

$$\lim_{x \rightarrow 0} \frac{e^x - P_4(x)}{x^5} = \text{constant}$$

In this example, the constant can be roughly estimated by $(0.01) \cdot (0.85)$. Where did this value come from?

- d. More generally, $f(x)$ has the same order of magnitude as $h(x)$ as $x \rightarrow a$ if there exists a constant c such that

$$\lim_{x \rightarrow a} \frac{f(x)}{h(x)} = c.$$

The notation for this statement is

$$f(x) = O(h(x)) \text{ as } x \rightarrow a$$

where the big O denotes order.

5. Let's get back to our main question: how well does the fourth degree Taylor polynomial $P_4(x)$ approximate the function $g(x)$ near $x = 0$? We found that the error function and a fifth degree function had the same order of magnitude as $x \rightarrow 0$:

$$[g(x) - P_4(x)] = O(x^5).$$

- a. Compare this statement with the definition of the order of magnitude on the previous page (part d). What is the function f ? What is the function h ? What is the limit point a ?

- b. From $g(x) - P_4(x) = O(x^5)$, we can write

$$e^x = \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}\right) + O(x^5).$$

Double check the definitions of g and P_4 from the first page.

Conclusion:

This worksheet has presented an example of Taylor's Theorem: If a function $f(x)$ has its first n derivatives defined at a point $x = a$, then

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k + R(x)$$

where $R(x)$ is the remainder and

$$R(x) = O([x-a]^{n+1}) \text{ as } x \rightarrow a.$$

From this theorem, we can say: since $g(x) = e^x$ has its first $n = 4$ derivatives defined at the point $x = 0$, then

$$g(x) = \sum_{k=0}^4 \frac{g^{(k)}(0)}{k!} (x-0)^k + R(x)$$

where $R(x)$ is the remainder and

$$R(x) = O([x-0]^{4+1}) = O(x^5) \text{ as } x \rightarrow 0.$$

