Consider the IVP model for the linear harmonic oscillator

\[ x' = v \quad x(0) = a \]
\[ v' = -b^2 x \quad v(0) = p \]

and the expression

\[ E = \frac{1}{2} (v(t))^2 + \frac{1}{2} b^2 (x(t))^2 \]

1. Show that \( \frac{dE}{dt} = 0 \) for all time \( t \)

2. What is the constant value of \( E \)?

\( E \) is known as a **first integral** for our given IVP. It is a combination of the variables involved in the model which have the composite property that they remain constant with respect to time. Oftentimes, first integrals of a mathematical model have a meaningful physical interpretation. In the above case, \( E \) represents the amount of energy (kinetic and potential) found in the oscillating system.

We have seen that first integral \( \alpha R - b \ln S = K \) exists for the **SIR model**

\[ S' = -aSI \]
\[ I' = aSI - bI \]
\[ R' = bI \]

and \( K = dR + bF - a \ln F - e \ln R \) for the **Lotka-Volterra Predator-Prey model**

\[ R' = aR - bRF \]
\[ F' = dRF - eF \]
Proving Periodicity of a solution of an IVP from a First Integral

CiC, p. 403 # 16(a). Show that the function \( E = \frac{1}{2}v^2 + \frac{25}{2} \ln(1 + x^2) \) is a first integral of the soft spring IVP model given by
\[
\begin{align*}
x' &= v \\
v' &= -\frac{25x}{1 + x^2} \quad x(0) = a \\
v(0) &= p
\end{align*}
\]

1. If the initial amplitude is \( a = 4 \) cm and the initial velocity is 0 cm/sec, what is the speed of the weight as it moves past the rest position?

2. Let’s consider all the stages of the soft spring. When does the spring achieve its maximum displacement? When does it achieve its maximum speed?

Is the motion periodic?