Class 19: Monday October 21

Another Application of Riemann Sums

Arc Length

Building the Sum. We have used Riemann sums to calculate areas, as interpretations of accumulated probability (in class) and accumulations of power with time (energy), of velocity with time (distance) from CiC. We can also use sums to calculate the length of a curve.
As we used areas of rectangles to approximate the area under a curve, we use the lengths of straight line segments to approximate the length of a curve. Picture the curve broken into n pieces, and consider a small piece of a curve between points (x_{k-1}, y_{k-1}) and (x_k, y_k).

Use the Pythagorean Theorem to find the distance between these two points, which we will denote by D_k.

$$\text{Distance}_k = D_k = \underline{\hspace{8cm}}.$$

This distance approximates the length of the curve between these two points. How could we improve the approximation?

We have approximated the length of one small piece of the curve. To approximate the entire length of the curve, we sum the approximations of the individual pieces:

$$\sum_{k=1}^{n} D_k = D_1 + D_2 + D_3 + \ldots + D_n.$$

As written, is this sum a **Riemann Sum**?

An Application. Suppose you are flying a kite such that the string is hanging in a parabolic arc. If we establish a coordinate system with the origin at your hand, then the height y, measured in feet, of the string at a distance x feet away from you is given by $y = x^2/20$. And the height of your kite is 245 feet. Sketch the kite and its string in a coordinate system, noting the coordinates of the endpoints of the kite string.

Suppose we partition the x interval into n equal subintervals, each of length Δx.

Now consider your equation for D_k. In it, there should be the terms $(y_k - y_{k-1})$. Define $(y_k - y_{k-1})$ as Δy and replace $x_k - x_{k-1}$ with Δx.

$$D_k = \underline{\qquad\qquad\qquad\qquad\qquad}$$

Use some algebra to rearrange D_k so that it looks like a Riemann Sum, i.e.

$$D_k = (\underline{\qquad\qquad\qquad\qquad\qquad}) \cdot \Delta x$$

If the number of subintervals n is set at 7, what is the size of Δx?

Use your formula for D_k with 7 subintervals to approximate the length of the kite string.

If we use many subintervals, i.e. $n \to \infty$, Δx approaches zero. How would our expression for D_k change?

In D_k there should be an expression that looks like $\dfrac{\Delta y}{\Delta x}$. How can you express the limit of this component in terms of a function of x? (Remember we know that $y = x^2/20$)

Write down an expression for the exact length of the kite, $D = \lim_{n\to\infty} \sum_{k=1}^{n} D_k$

Can you find D exactly? If not, how would you estimate this value?