Class 15: Monday October 7

Summary of Multivariable Optimization and Introduction to Constrained Optimization

Optimization Summary

The extreme values of $f(x, y)$ can only occur at

(i) boundary points of the domain of f.

(ii) critical points of f, i.e. **interior points** where $f_x = f_y = 0$ simultaneously, or points where f_x or f_y fails to exist.

If the first and second derivatives (f_x, f_y, f_{xx}, f_{yy} and f_{xy}) are continuous through an open region containing a point (a, b) where $f_x(a, b) = f_y(a, b) = 0$ you can classify the critical point (a, b) using the **Second Derivative Test**:

(i) $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at $(a, b) \Longrightarrow$ **LOCAL MAXIMUM**.

(ii) $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at $(a, b) \Longrightarrow$ **LOCAL MINIMUM**.

(iii) $f_{xx}f_{yy} - f_{xy}^2 < 0$ at $(a, b) \Longrightarrow$ **SADDLE POINT**.

(iv) $f_{xx}f_{yy} - f_{xy}^2 = 0$ at $(a, b) \Longrightarrow$ **NO CONCLUSION**!

Examples

1. Find the absolute max and min of the function $f(x, y) = xy$

2. **CiC, 521, #13a.** Find the extrema of $f(x, y) = 3x^2 + 7xy + 2y^2 + 5x - 6y + 3$

Constrained Optimization

So far we have only considered the formula for a function we wish to optimize. But just as in functions of one variable, the *domain* of a function of two variables is very important in optimization. The domain is often specified in the form of a *constraint*.

Examples

3. Determine the *extrema* of $f(x,y) = xy$ subject to the constraints

$$x \geq 0, \quad , y \geq 0, \quad 3x + 8y \leq 120$$

To help you solve this problem, first sketch the boundary of the constraint set.

Evaluate $f(x,y)$ along the boundary when $x = 0$

Evaluate $f(x,y)$ along the boundary when $y = 0$

Let $3x + 8y = 120$, solve for y and obtain an expression $f(x,y) = A(x)$ which we maximize on the domain $x \geq 0$.

Compare values of $f(x, y$ found along the boundary and obtain the extrema that way.

4. **CiC, 521, #11.** $f(x,y) = x^2y$ where $x + 5y = 10$ and $x \geq 0$ and $y \geq 0$