Class 13: Wednesday October 2

Multivariable Linear Functions and Multivariable Local Linearity

Linear Functions of Two Variables

Definition: $z = f(x, y)$ is a **linear** function of two variables if it satisfies:

$$\begin{aligned} \Delta z &= p\Delta x + q\Delta y, \qquad\qquad p \text{ and } q \text{ constant} \\ z - z_0 &= p(x - x_0) + q(y - y_0) \end{aligned}$$

The definition of a linear function of two variables is just an extension of the definition of a linear function of one variable. Many other features of linear functions of one and two variables are also similar:

	One Variable	*Two Variables*
Initial Value Form:	$\Delta y = m\Delta x$ $y - y_0 = m \cdot (x - x_0)$	$\Delta z = p\Delta x + q\Delta y$ $z - z_0 = p \cdot (x - x_0) + q \cdot (y - y_0)$
Intercept Form:	$y(x_0) = y_0$ $y = mx + b$	$z(x_0, y_0) = z_0$ $z = px + qy + r$
Constant Rates of Change:	$dy/dx = m$	$\partial z/\partial x = p, \quad \partial z/\partial y = q$
	graph is a line	surface graph is a plane
Plots:		contour plot with equally spaced levels has (equally spaced) parallel lines provided not both $p = 0$, $q = 0$. slope of lines in contour plot is $\Delta y/\Delta x = -p/q$

<u>Example</u>

1. Write the linear function satisfying $\Delta z = 4\Delta x + 4\Delta y$, $z(2, 2) = 8$, in intercept form.

Local Linearity

Definitions: $z = f(x, y)$ is *locally linear* at $(x, y) = (x_0, y_0)$ if the surface graph approaches a plane as you zoom in on the point $(x_0, y_0, f(x_0, y_0))$. This plane is called the *tangent plane* to the graph at this point. The equation of this plane *approximates* the function near (x_0, y_0). When comparing changes on the the tangent plane with changes on the function graph, we often write increments on the tangent plane as *differentials*.

Local linearity for a function of two variables is an extension of this concept for a function of one variable:

	One Variable $dy = mdx$	*Two Variables* $dz = pdx + qdy$
Equation for Tangent: (Initial Value Form)	$y - y_0 = m \cdot (x - x_0)$ $m = f'(x_0)$ $y_0 = f(x_0)$	$z - z_0 = p \cdot (x - x_0) + q \cdot (y - y_0)$ $p = f_x(x_0, y_0), q = f_y(x_0, y_0)$ $z_0 = f(x_0, y_0)$
Microscope Approximation:	$\Delta y \approx m\Delta x$ $f(x) - y_0 \approx m \cdot (x - x_0)$	$\Delta z \approx p\Delta x + q\Delta y$ $f(x, y) - z_0 \approx p \cdot (x - x_0) + q \cdot (y - y_0$
Zooming In:	graph approaches tangent line	surface graph approaches tangen plane contour plot approaches contour plo of tangent plane (if tangent plane not horizontal) slope of line tangent to level curve a (x_0, y_0): $dy/dx = -p/q$ $dy/dx = -f_x(x_0, y_0)/f_y(x_0, y_0)$

<u>Example</u>

2. $z = f(x, y) = x^2 + y^2$ is locally linear at $(x_0, y_0) = (2, 2)$. Write the equation, in initial value form, of the plane tangent to the surface graph of f at $(x_0, y_0, z_0) = (2, 2, 8)$. Also write the Microscope Approximation for f at about the point $(x_0, y_0) = (2, 2)$. Compare $f(2.01, 1.95)$ with the approximate value obtained using the Microscope Approximation.