Class 6: Monday September 16

Successive approximations and the formal definition of limit

Warm-up:
Using our model, some initial values for S, I, and R, and different values of Δt in the `Excel` spreadsheet some predictions about the future population sizes after 3 days can be made.

$$\begin{aligned} S'(t) &= -.00001S(t)I(t) \\ I'(t) &= .00001S(t)I(t) - 1/14I(t) \\ R'(t) &= 1/14I(t) \end{aligned}$$

$$S(0) = 45,400 \qquad I(0) = 2100 \qquad R(0) = 2500.$$

Estimated values of S, I, and R when $t = 3$

Δt	$S(3)$	$I(3)$	$R(3)$
1	41,435.728	5422.138	3142.1341
0.1	40,578.415	6121.2529	3300.3317
0.01	40,472.189	6207.2900	3320.5212
0.001	40,461.32	6216.0853	3322.5944
0.0001	40,460.231	6216.9668	3322.8023
0.00001	40,460.122	6217.0549	3322.8231

1. What do you suppose are the actual values of S, I and R at $t = 3$?

2. Why are you confident that these are the actual values?

We know that **Euler's Method** is a numerical technique for approximating solutions to initial value problems. If we decrease the size of the time step, we know that the error Euler makes decreases. Let's show exactly *how* the Euler error is related to the time step Δt.
Recall the IVP from homework #2:

$$y' = 0.321t^2, \qquad y(0) = 0.41$$

Here are the results for using Euler's Method with increasingly smaller Δt to approximate $y(1)$. The exact solution to the IVP was $y(t) = 0.107t^3 + 0.41$

Δt	$\tilde{y}(1)$	Absolute Error
1	0.41	0.1070
0.5	0.4501	0.0669
0.25	0.4801	0.0369
0.1	0.5015	0.0155
0.01	0.5154	0.0016
0.001	0.5168	0.0002

1. When the size of Δt decreases by a factor of 2, what happens to the absolute error E?

2. When the size of Δt decreases by a factor of 10, what happens to the absolute error E?

3. Can you write an expression for the relationship between Euler error, E and the time step used, Δt?

The concept of LIMITS: The concept of a *limit* is a central theme of all Calculus courses. When using the Babylonian Algorithm to estimate $\sqrt{a}$, the sequence of successive approximations

$$\begin{aligned} x_1 & \\ x_2 &= \frac{x_1 + \frac{a}{x_1}}{2} \\ x_3 &= \frac{x_2 + \frac{a}{x_2}}{2} \\ &\vdots \\ x_n &= \frac{x_{n-1} + \frac{a}{x_{n-1}}}{2} \\ &\vdots \end{aligned}$$

stabilizes. We say that the *limit* of the sequence of successive approximations is $\sqrt{a}$, or, rather, that

$$\lim_{n \to \infty} x_n = \sqrt{a}.$$

Formal Definition

The limit L of a sequence x_n is said to exist if *for any* ϵ there exists a number N so that when $n > N, \quad |x_n - L| < \epsilon$.

What does this mean? (Can you draw a picture of the concept of limit of a sequence)?

Think-Pair-Share Exercise

Write down a sentence **in your own words** expressing the definition of a limit of a sequence.

Then share your definition with your neighbor.

Babylonian Algorithm Error

The following is a sequence generated by the Babylonian algorithm as it attempts to approximate $\sqrt{17} = 4.123105626$.

Step n	Approximation x_n	Absolute Error
1	1.000000	
2	9.000000	
3	5.444444	
4	4.283446	
5	4.126107	
6	4.12310	

Can you write down an expression for the error in the current Babylonian approximation related to the error in the previous Babylonian approximation?