Math 118 – Homework #17 SOLUTION (3 points)

CiC, p. 402-3, # 13.

E is a first integral because it is constant for all t. We can see this by showing that $E' = 0$:

$$\begin{aligned} E'(t) &= \frac{1}{2}(2v \cdot v') + \frac{1}{2}b^2(2x \cdot x') + \frac{1}{4}\beta(4x^3 \cdot x') \\ &= v(-b^2x - \beta x^3) + b^2xv + \beta x^3 v \\ &= 0 \end{aligned}$$

CiC, p. 403, # 14.

Given: $x(0) = 2,\ v(0) = 0,\ b = 4,\ \beta = 1.$

a. The value of E is

$$E(0) = \frac{1}{2}[v(0)]^2 + \frac{1}{2}b^2[x(0)]^2 + \frac{1}{4}\beta[x(0)]^4 = 36$$

b. At the rest position, $x = 0$. To find v, we use the constant value of E:

$$36 = E = \frac{1}{2}v^2 + \frac{1}{2}b^2[0]^2 + \frac{1}{4}\beta[0]^4 = \frac{1}{2}v^2$$

So $v = \pm\sqrt{72} = \pm 8.485$ cm/sec.

c. If $x > 2$, then the sum of the last two terms of the first integral $(\frac{1}{2}b^2x^2 + \frac{1}{4}\beta x^4)$ is greater than $32 + 4 = 36$. Since the first term of the first integral is never negative, we would have $E > 36$ if $x > 2$. Since E is always equal to 36, x cannot be greater than 2.

CiC, p. 403, # 15.

From Exercise #13, we know that

$$E(t) = \frac{1}{2}[v(t)]^2 + \frac{1}{2}b^2[x(t)]^2 + \frac{1}{4}\beta[x(t)]^4$$

is a first integral of the system of differential equations.

Using the initial conditions: $x(0) = a,\ v(0) = p$, we can determine the total energy in the system:

$$E(0) = \frac{1}{2}[p]^2 + \frac{1}{2}b^2[a]^2 + \frac{1}{4}\beta[a]^4 = \frac{p^2}{2} + \frac{a^2b^2}{2} + \frac{a^4\beta}{4}.$$

Using this constant, we can show that whenever $x = a$, we have $v = \pm p$ which implies that we return to our initial conditions $x = a, v = p$ so that the cycle repeats. Let $x = a$ in E:

$$\frac{p^2}{2} + \frac{a^2b^2}{2} + \frac{a^4\beta}{4} = E = \frac{1}{2}v^2 + \frac{1}{2}b^2a^2 + \frac{1}{4}\beta a^4$$

Then $p^2 = v^2$ so that $v = \pm p$.