Final Exam: Math 118

Advanced Placement Calculus — Wednesday, December 11, 2002

Name: ______________________________

Directions:

- STOP!! READ THIS!! IT IS WRITTEN FOR YOU!!
- Write your name on the line above if you have not done so already.
- Sign your name on the pledge below if you have not done so already.
- You should work through these pages at a steady pace.
- Remember to show your work. If you got the correct answer without showing any work, you will not receive full credit. If you got the incorrect answer while showing your work, you may receive substantial partial credit.
- Keep 4 decimal places of accuracy where appropriate.
- If explanations are asked for, write in complete sentences.
- **Write down on the this page which EIGHT of the TEN problems you want graded. ONLY THE PROBLEMS LISTED HERE WILL BE GRADED.**
- Ok, now then. You have a lot of time, so relax. Remember to breathe.

Pledge: I, ____________________________, pledge my honor as a human being and Occidental student, that I have followed all the rules above to the letter and in spirit.

No.	Score	Maximum
		25
		25
		25
		25
		25
		25
		25
		25
Total		**200**

1. Consider the functions

$$f(x) = \int_0^x g(t)dt \text{ and } F(x) = \int_0^{f(x)} g(t)dt$$

where $\int_0^1 g(t)dt = 2$ and $g(0) = -1$, $g(1) = 0.5$ and $g(2) = 1$.

For each of the following expressions obtain an exact numerical value. Show your work.

a. (5 points) $f(1)$

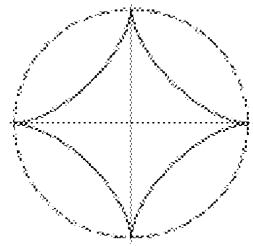
b. (5 points) $F(0)$

c. (5 points) $f'(0)$

d. (5 points) $F'(0)$

e. (5 points) $F'(1)$

2. Consider the graph of the 4-cusped hypocycloid (ok, just a fancy name for a special kind of curve) known as an **astroid** below. It is symmetric about both the x and y axes.



The equation of the curve is $x^{2/3} + y^{2/3} = 1$. We want to obtain the length L of the entire astroid found inside the unit circle.

a. (5 points) Rewrite the equation of the astroid so that it looks like $y = f(x)$.

b. (5 points) What are the coordinates of the points where the curve intercepts the positive x-axis and the positive y-axis?

c. (10 points) Use your previous answers and the integral formula for the length of a curve to show that the length of the **entire** astroid is exactly $\lim_{a \to 0+} 4 \int_a^1 \frac{1}{x^{1/3}} dx = 4 \int_0^1 \frac{1}{x^{1/3}} dx = L$

d. (5 points) What is the value of the total length L of the astroid?

3. Two students are feverishly studying for their Calculus Final and have a disagreement about the initial value problem $y' = f(x, y), \qquad y(a) = b$. Below is an excerpt of their conversation:

Sydney: **I don't know why we learned all those other methods of solving initial value problems. The only technique that really always seems to work is separation of variables.**

Vaughn: **You doofus! The only method that always works to solve an initial value problem is Euler's Method. This method only finds an approximation to the solution $y(x)$ which solves the initial value problem, but since we know how to improve the approximation, this is really the only method we need to know.**

Sydney: **The professor himself says that Euler's Method sucks! You don't even get an explicit formula for the exact solution $y(x)$ to the initial value problem. With separation of variables I am using the Fundamental Theorem of Calculus to get my solution, while you are using a silly inaccurate algorithm, so my method *must* be better.**

Vaughn: **Well, it's not always true that you can find an explicit form of the solution to the initial value problem. Euler's Method is related to first-order Taylor polynomials and the Miscroscope Approximation and we both know that these concepts are also central to Calculus.**

Write **at least 5 sentences** discussing which student you think has a better understanding of Calculus. Identify **any** and all correct, incorrect or partly correct statements made by the students. If a statement is incorrect explain why. **You must be careful not to make any incorrect statements yourself in your explanation.** PROOFREAD YOUR ANSWER.

4. Consider the surface $z = f(x, y) = (xy)^2 = x^2y^2$

(a) *(5 points)* Find the following derivatives:

$f_x =$

$f_y =$

$f_{xx} =$

$f_{yy} =$

$f_{xy} =$

(b) (7 points) Find *all* the critical points of f.

(b) (6 points) Write down the absolute (global) maximum value and minimum value of the function $f(x, y)$ (if they exist!).

(c) (7 points) Now suppose the input values to the function $f(x, y)$ are constrained to the circle of unit radius at the origin and its interior: $x^2 + y^2 \leq R^2$. Find the absolute maximum value of $f(x, y)$ subject to this constraint.

5. Consider the initial value problem $x'' = -16x$ with $x(0) = 9$ and $x'(0) = 16$

a. (10 points) Write down the exact solution to the initial value problem. (Check your answer!)

b. (10 points) A **boundary value problem** is identical to an initial value problem but instead of insisting that the solution and its derivative have a certain value at a single point (i.e. $x(0)$ and $x'(0)$ above) the solution of a boundary value problem must have certain values at two different points. Knowing this, show that solution to part (a) solves the boundary value problem $x'' = -16x$ with $x(0) = 9$ and $x(\pi/4) = -9$.

c. (5 points) Write down another boundary value problem that the solution to the initial value problem in part (a) and boudary value problem in part (b) ALSO solves.

6. Do the following series converge? Why or why not? Be sure to indicate which test you use to support your statement of convergence and divergence.

a. (8 points) $\displaystyle\sum_{n=1}^{\infty} \frac{9}{n^{5/3}}$

b. (9 points) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{3^n}$

c. (8 points) $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln n)^n}{e^n}$

7. Consider the fourier series for the mystery function $f(x)$

$$f(x) = \frac{1}{3} + \sum_{k=1}^{\infty} (-1)^k \frac{4}{k^2\pi^2} \cos(k\pi x)$$

Use the information contained in the Fourier Series to answer the following questions about $f(x)$. **EXPLAIN YOUR ANSWERS.**

a. (*5 points*) What is the period of the mystery function $f(x)$?

b. (*5 points*) What is the average value of the mystery function $f(x)$ on each period?

c. (*5 points*) Is the mystery function even or odd or neither?

d. (*10 points*) Given the information about the mystery function that $f(1) = 1$ show that this implies that $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$. (HINT: write out the first few terms of the right hand side of $f(1)$)

8. Consider the Taylor Series for the function $\ln(t)$ about the point $t = 1$ given below

$$\ln t = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(t-1)^k}{k}$$

a. (6 points) When $t = 0$ show that at this point the Taylor series is simply a multiple of the harmonic series. Does this series converge or diverge? (PROVE IT!)

b. (2 points) From your answer above, is $t = 0$ in the interval of convergence of the Taylor Series for $\ln t$? In other words, at $t = 0$, is the function value $\ln 0$ equal to whatever the Taylor series converges (if it converges) to when $t = 0$?

c. (6 points) When $t = 2$ show that at this point the Taylor series is simply the **alternating harmonic series**. Does this series converge or diverge? (PROVE IT!)

d. (2 points) From your answer above, is $t = 2$ in the interval of convergence of the Taylor Series for $\ln t$? In other words, at $t = 2$, is the function value $\ln 2$ equal to whatever the Taylor series converges (if it converges) to when $t = 2$?

e. (9 points) Use the **Absolute Ratio Test** on the infinite series $\sum_{k=1}^{\infty} a_k$ where $a_k = (-1)^{k+1} \frac{(t-1)^k}{k}$ to find conditions on t for which the infinite series converges.

9. The potential energy, $U(r)$, in a diatomic molecule is given by the formula

$$U(r) = A\left[\left(\frac{b}{r}\right)^{12} - 2\left(\frac{b}{r}\right)^{6}\right]$$

where A and b are **positive constants** and r is the distance *between* the two atoms. For which value of r is $U(r)$ a minimum?

What is the minimum value of $U(r)$?

10. Consider the following IVP.

$$y' = \ln t \qquad\qquad y(1) = 3.$$

Your goal is to find the value of $y(2)$ to 2 decimal places. Do this in atleast 4 possible ways, showing your work and explaining your methods clearly. If your method is not giving you 2 places of accuracy, explain what you would have to do to INSURE that you would be able to get 2 decimal places of accuracy.