2A: 6. \( f(x) = (x^3 - 2x^2 + 5)(x^4 - 3x^2 + 2) \)

Using the product rule in combination with the power rule:

\[
f'(x) = (x^3 - 2x^2 + 5)(4x^3 - 6x) + (3x^2 - 4x)(x^4 - 3x^2 + 2)
\]

Simplifying:

\[
= 4x^6 - 8x^5 + 20x^3 - 6x^4 + 12x^4 - 30x
\]

\[
= 12x^3 - 9x^4 + 6x^2 - 4x^6 + 12x^3 - 8x
\]

\[
f(x) = 4x^6 - 12x^5 - 15x^4 + 44x^3 + 6x^2 - 38x
\]

19. \( f(x) = \frac{x^2 + 3x - 2}{\sqrt{x}} = \frac{x^2 + 3x - 2}{x^{1/2}} \)

There are 2 ways to take this derivative. You can use the quotient rule (i) or simplify (ii):

(i) Using the quotient and power rules:

\[
f'(x) = \frac{(x^{1/2})(2x + 3) - (x^2 + 3x - 2)(\frac{1}{2}x^{-1/2})}{(x^{1/2})^2}
\]

\[
= \frac{2x^{3/2} + 3x^{1/2} - (\frac{1}{2}x^{3/2} + \frac{3}{2}x^{1/2} - x^{-1/2})}{x}
\]

\[
= \frac{2x^{3/2} + 3x^{1/2} - \frac{1}{2}x^{3/2} - \frac{3}{2}x^{1/2} + x^{-1/2}}{x^{1/2}}
\]

\[
= \frac{2x^2 + 3x - \frac{1}{2}x^2 - \frac{3}{2}x + 1}{x^{3/2}}
\]

(continued on next page)
Continuing from previous page:

\[
\frac{3}{a} x^2 + \frac{3}{a} x + 1 = \frac{\frac{3}{a} (x^2)}{x^{3/2}} + \frac{\frac{3}{a} (x)}{x^{3/2}} + \frac{1}{x^{3/2}}
\]

\[
f'(x) = \frac{3}{2} x^{1/2} + \frac{3}{2} x^{-1/2} + x^{-3/2}
\]

(ii) Simplifying first:

\[
\frac{x^2 + 3x - 2}{x^{1/2}} = \frac{x^2}{x^{1/2}} + \frac{3x}{x^{1/2}} - \frac{2}{x^{1/2}} = x^{3/2} + 3x^{1/2} - 2x^{-1/2}
\]

Now using the power rule:

\[
f'(x) = \frac{3}{2} x^{1/2} + 3 \cdot \frac{1}{2} x^{-1/2} - 2 \left(- \frac{1}{2}\right) x^{-3/2}
\]

\[
f'(x) = \frac{3}{2} x^{1/2} + \frac{3}{2} x^{-1/2} + x^{-3/2}
\]

20. \( f(x) = \frac{2x}{x^2 + 1} \)

Using the quotient rule and power rule:

\[
f'(x) = \frac{(x^2 + 1)(2) - (2x)(2x)}{(x^2 + 1)^2} = \frac{2x^2 + 2 - 4x^2}{(x^2 + 1)^2} = -\frac{2x^2 + 2}{(x^2 + 1)^2}
\]

\[
f'(x) = -\frac{2x^2 + 2}{(x^2 + 1)^2}
\]
25. We have a "big" function: \( \lambda(x) = f(x)g(x)h(x) \)
Let's really \( f(x)g(x) = p(x) \). Then \( \lambda(x) = p(x) \cdot h(x) \).
According to the product rule:
\[
\lambda'(x) = p(x) \cdot h'(x) + p'(x) \cdot h(x)
\]
But, also according to the product rule (and the definition
\( p(x) = f(x)g(x) \)):
\[
p'(x) = f(x)g'(x) + f'(x)g(x)
\]
So, substituting:
\[
\lambda'(x) = f(x)g(x)h'(x) + \left[ f(x)g'(x) + f'(x)g(x) \right] h(x)
\]
So:
\[
\lambda'(x) = f(x)g(x)h'(x) + f(x)g'(x)h(x) + f'(x)g(x)h(x).
\]
In general,
\[
\left[ f_1(x) f_2(x) f_3(x) \ldots f_n(x) \right]' = \\
= f_1'(x) f_2(x) f_3(x) \ldots f_n(x) \\
+ f_1(x) f_2'(x) f_3(x) \ldots f_n(x) \\
+ \ldots \\
+ f_1(x) f_2(x) f_3'(x) \ldots f_n(x)
\]
That is, the derivative is the sum of \( n \) terms in each of which the \( n \)th function is differentiated.
26. \((g(x))^{-1} = \frac{-1}{g(x)}\)

So, according to the quotient rule:
\[
\left[\frac{-1}{g(x)}\right]' = \frac{g(x) \cdot 0 + g'(x)(-1)}{(g(x))^2} = \frac{-g'(x)}{(g(x))^2} = -g'(x)(g(x))^{-2}
\]

According to the product rule:
\[
\left[f(x)(g(x))^{-1}\right]' = f(x) \{ (g(x))^{-2}g' + f'(x)(g(x))^{-1} \}
\]

Then using the first result and substituting it in:
\[
= f(x) \left\{ -g'(x)(g(x))^{-2}g' + f'(x)(g(x))^{-1} \right\}
\]

\[
\left[f(x)g(x)\right]' = \frac{-f(x)g'(x)}{(g(x))^2} + \frac{f'(x)}{g(x)}
\]

45. \(F(x) = f(x)g(x)\)

\(F'(x) = f'(x)g(x) + f(x)g'(x)\), according to the product rule.

Using the product rule again (twice now!), we obtain:
\(F''(x) = f''(x)g(x) + f(x)g''(x) + f'(x)g'(x)\)

\(F''(x) = f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x)\)

We know \((a+b)^2 = a^2 + 2ab + b^2\). If, in this form, the power represents the derivative (i.e., 1 \(\Rightarrow\) first derivative and 2 \(\Rightarrow\) second derivative) and \(a = f(x)\), \(b = g(x)\) then \(F''(x)\) matches the form \((a+b)^2\).
Using the product rule three times (once per term) on \( F''(x) \):

\[
F''(x) = f''(x)g'(x) + f''(x)g(x) + 2\left( f'(x)g''(x) + f''(x)g'(x) \right) \\
+ f(x)g'''(x) + f'(x)g''(x) \quad \text{[since the derivative of } c \cdot f(x) \text{ is } c \cdot f'(x)]
\]

\[
F''(x) = f''(x)g(x) + 3f''(x)g'(x) + 3f'(x)g''(x) + f(x)g'''(x)
\]

We know \((a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3\). Using powers to denote the derivative and \(a = f(x), b = g(x)\) as before, \( F''(x) \) matches the form of \((a+b)^3\).

47. \( g(x) = [f(x)]^2 = f(x) \cdot f(x) \)

According to the product rule,

\[
g'(x) = f(x) \cdot f'(x) + f'(x) \cdot f(x) \]

\[
g'(x) = 2f(x)f'(x)
\]

This is the desired form.

D. 7: 17. \( \sin^3x \)

Let \( f(g) = g^3 \) and \( g(x) = \sin x \)

Then \( f \circ g(x) = f(g(x)) = f(\sin x) = (\sin x)^3 = \sin^3x \)

26. \( \ln 3x - 5 \)

Let \( f(g) = g - 5 \) and \( g(x) = \ln 3x \)

Then \( f \circ g(x) = f(g(x)) = f(\ln 3x) = \ln 3x - 5 \)
2.7: 1. Fred's rate of motion: \( F' = 10 \text{ mph} \)
Greg's rate of motion: \( G' = 2F' \)

Or, in other words, Greg can run twice as fast as Fred, so that \( G = 2F \) and \( G' = 2F' \)

In this context, the chain rule is obvious.

31. \( f(x) = \sin(\ln(\cos x^3)) \)

\[
f'(x) = \cos(\ln(\cos x^3)) \cdot \left( \frac{1}{\cos x^3} \right) \cdot (-\sin x^3) \cdot (3x^2)
\]

\[
f'(x) = -3x^2 \cdot \tan x^3 \cdot \cos(\ln(\cos x^3))
\]

38. \( f(x) = \sqrt{\frac{x \sin x}{x^2 + 4}} = (\frac{x \sin x}{x^2 + 4})^{1/2} \)

\[
f'(x) = \frac{1}{2} \left( \frac{x \sin x}{x^2 + 4} \right)^{-1/2} \left( \frac{\frac{x^2 + 4^2(x \cos x + \sin x) - x \sin x (2x)}{(x^2 + 4)^2}}{x^2 + 4} \right)
\]

\[
\text{chain rule} \quad \text{quotient rule w/ product \& power rule}
\]

\[
= \frac{1}{2} \sqrt{\frac{x^2 + 4}{x \sin x}} \left\{ \frac{x^3 \cos x + x^2 \sin x + 4x \cos x + 4 \sin x - 2x^2 \sin x}{(x^2 + 4)^2} \right\}
\]

\[
f'(x) = \frac{1}{2} \frac{x^3 \cos x - x^2 \sin x + 4x \cos x + 4 \sin x}{(x \sin x)(x^2 + 4)^{3/2}}
\]
2. In this problem, the derivative is being taken with respect to \( x \), i.e., \( y' \) means \( \frac{d}{dx} (y) \). Instead of thinking of implicit differentiation think of the chain rule. Then, differentiating both sides:

\[
\frac{d}{dx} (x^2 y^2 + 3) = \frac{d}{dx} (x)
\]

(and using product rule as well)

\[
x^2 \cdot \frac{d}{dx} (y^2) + \frac{d}{dx} (x^2) y^2 + \frac{d}{dx} 3 = \frac{1}{dx} x
\]

\[
= x^2 (2y \frac{dy}{dx}) + 2x \frac{dx}{dx} y^2 + 0 = \frac{dx}{dx}
\]

But \( \frac{dx}{dx} = 1 \):

\[
= 2x^2 y \frac{dy}{dx} + 2xy^2 = 1
\]

So it appears that we "tack on" a \( y' \) and take "regular" derivatives of \( x \) while in fact we're taking all "regular" derivatives with the chain rule, \( \frac{d}{dx} \) goes away since \( \frac{dx}{dx} = 1 \) and \( \frac{dy}{dx} \) "stays around" since we don't know what \( \frac{dy}{dx} \) is.

12. \( \sin xy = x^2 - 3 \)

\[
\frac{d}{dx} (\sin xy) = \frac{d}{dx} (x^2 - 3)
\]

Differentiate both sides w.r.t. \( x \)

\[
\cos xy \left( \frac{d}{dx} x + y \cdot \frac{dy}{dx} \right) = 2x \frac{dx}{dx} - 0
\]

Apply chain and product rules.

\[
\cos xy \left( 1 \cdot y + x \cdot \frac{dy}{dx} \right) = 2x
\]

Simplify

(continued next page)
\[ y \cdot \cos xy + x \cdot \cos xy \frac{dy}{dx} = 2x \quad \text{Simplify} \]

\[ y' = \frac{dy}{dx} = \frac{2x - y \cdot \cos xy}{x \cdot \cos xy} \quad \text{Solve for} \quad \frac{dy}{dx} = y' \]

20. \[ e^{x^2}y - 3y = x^2 + 1 \]

\[ \frac{d}{dx}(e^{x^2}y - 3y) = \frac{d}{dx}(x^2 + 1) \quad \text{Differentiate both sides of the equation wrt x} \]

\[ e^{x^2} \frac{dy}{dx} + e^{x^2} \cdot 2x \cdot \frac{dx}{dx} \cdot y - 3 \frac{dy}{dx} = \frac{d}{dx}(x^2) + \frac{d}{dx}(1) \quad \text{"distribute" the derivative} \]

\[ e^{x^2} \frac{dy}{dx} + e^{x^2} \cdot 2x \cdot \frac{dy}{dx} \cdot y - 3 \frac{dy}{dx} = 2x \cdot \frac{dx}{dx} + 0 \quad \text{Simplify} \]

\[ e^{x^2} \frac{dy}{dx} + 2xye^{x^2} - 3 \frac{dy}{dx} = 2x \]

\[ (e^{x^2} - 3) \frac{dy}{dx} = 2x (1 - ye^{x^2}) \]

\[ y' = \frac{dy}{dx} = \frac{2x (1 - ye^{x^2})}{e^{x^2} - 3} \]