5. \( f(x) = 3x + 1, \ a = 1 \)

Two ways:
(a) \[ f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{3(a+h) + 1 - (3a+1)}{h} = \lim_{h \to 0} \frac{3a + 3h + 1 - 3a - 1}{h} = \lim_{h \to 0} \frac{3h}{h} = \lim_{h \to 0} 3 = 3 \]

\[ f'(a) = 3 \Rightarrow f'(1) = 3 \]

OR
(b) \[ f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{3(1+h) + 1 - (3\cdot1+1)}{h} = \lim_{h \to 0} \frac{3h}{h} = \lim_{h \to 0} 3 = 3 \]

\[ f'(1) = 3 \]

6. \( f(x) = 3x^2 + 1, \ a = 1 \)

Two ways:
(a) \[ f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{3(a+h)^2 + 1 - (3a^2 + 1)}{h} = \lim_{h \to 0} \frac{3a^2 + 6ah + 3h^2 + 1 - 3a^2 - 1}{h} = \lim_{h \to 0} \frac{3a^2 + 6ah + 3h^2 + 1 - 3a^2 - 1}{h} = \lim_{h \to 0} \frac{6ah + 3h^2}{h} = 6a \]

\[ f'(a) = 6a \]

(b) \[ f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{3(1+h)^2 + 1 - (3\cdot1+1)}{h} = \lim_{h \to 0} \frac{3h^2 + 6h + 1}{h} = \lim_{h \to 0} (3h + 6) = 6 \]

\[ f'(1) = 6 \]

\[ \text{cont'd next page} \]
6 cont'd

\[ \lim_{h \to 0} \frac{6ah + 3h^2}{h} = \lim_{h \to 0} \frac{f'(6a + 3h)}{h} \]

\[ = \lim_{h \to 0} 6a + 3h = 6a \]

\[ f'(a) = 6a \Rightarrow f'(1) = 6 \]

OR

\[ f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{3(1+h)^2 + 1 - (3(1)^2 + 1)}{h} \]

\[ = \lim_{h \to 0} \frac{3(1 + 2h + h^2) + 1 - 3 - 1}{h} \]

\[ = \lim_{h \to 0} \frac{3 + 6h + 3h^2 - 3}{h} = \lim_{h \to 0} \frac{6h + 3h^2}{h} \]

\[ = \lim_{h \to 0} 6 + 3h = 6 \]

7. \( f(x) = \sqrt{3x+1} \); \( a = 1 \)

Two ways

(a) \( f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{\sqrt{3(a+h)+1} - \sqrt{3a+1}}{h} \)

\[ = \lim_{h \to 0} \frac{\sqrt{3a+3h+1} - \sqrt{3a+1}}{h} \cdot \frac{\sqrt{3a+3h+1} + \sqrt{3a+1}}{\sqrt{3a+3h+1} + \sqrt{3a+1}} \]

\[ = \lim_{h \to 0} \frac{(3a + 3h + 1) - (3a + 1)}{h [\sqrt{3a+3h+1} + \sqrt{3a+1}]} \]

\[ = \lim_{h \to 0} \frac{3h}{h [\sqrt{3a+3h+1} + \sqrt{3a+1}]} = \frac{3}{\sqrt{3a+3} + \sqrt{3a+1}} = \frac{3}{2 \sqrt{3a+1}} \]

\[ f'(a) = \frac{3}{2 \sqrt{3a+1}} \Rightarrow f'(1) = \frac{3}{2 \sqrt{3+1}} = \frac{3}{4} \]
57. \( f \rightarrow \) \( \rightarrow \) \( \rightarrow \) in \( 180^\circ \)

This should take \( \approx 180^\circ \). \( \text{ms} \) \( \frac{.0015}{1 \text{ ms}} \) \( \approx 0.016 \text{ s} \)

\( \rightarrow \) \( \rightarrow \) \( \rightarrow \) in \( 90^\circ \)

This should take \( \approx 90^\circ \). \( \text{ms} \) \( \frac{.0015}{1 \text{ ms}} \) \( \approx 0.008 \text{ s} \)

2.5: 6, 10

6. \( f(x) = x^2 + a \cos x \)

\( f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 + 2 \cos(x+h) - (x^2 + 2 \cos x)}{h} \)

\( = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + 2(\cos x \cos h - \sin x \sin h) - x^2 - 2 \cos x}{h} \)

\( = \lim_{h \to 0} \frac{2xh + h^2 + 2 \cos x \cos h - 2 \cos x - 2 \sin x \sin h}{h} \)

\( = \lim_{h \to 0} \frac{2xh}{h} + \lim_{h \to 0} \frac{2 \cos x (\cosh - 1)}{h} - \lim_{h \to 0} \frac{2 \sin x \sinh}{h} \)

\( = 2x + 2 \cos x \cdot 0 - 2 \sin x \cdot 1 = 2x - 2 \sin x \)

\[ f'(x) = 2x - 2 \sin x \]

10. \( f(x) = 4x^2 - 3 \tan x \)

\( f'(x) = \lim_{h \to 0} \frac{4(x+h)^2 - 3 \tan(x+h) - (4x^2 - 3 \tan x)}{h} \)

\( = \lim_{h \to 0} \frac{4(x^2 + 2xh + h^2) - 3 \tan(x+h) - 4x^2 + 3 \tan x}{h} \)

\( = \lim_{h \to 0} \frac{4x^2 + 2xh + h^2 - 4x^2}{h} + \lim_{h \to 0} \frac{3 \tan x - 3 \tan(x+h)}{h} \)

\( = \lim_{h \to 0} \frac{2xh + h^2}{h} + 3 \lim_{h \to 0} \frac{\tan x - \tan(x+h)}{h} \)
\[ \lim_{h \to 0} 2x + 3 \lim_{h \to 0} \frac{\sin x \cos (x+h) - \cos x \sin (x+h)}{h} = 2x + 3 \lim_{h \to 0} \frac{\sin x \cosh - \sin x \sinh - \sin x \cosh + \sinh \cos x}{h \cos x \cos (x+h)} = 2x + 3 \lim_{h \to 0} \frac{-\sin^2 x \sinh - \sinh \cos^2 x}{h \cos x \cos (x+h)} = 2x + 3 \lim_{h \to 0} -\sinh \frac{(\sin^2 x + \cos^2 x)}{h \cos x \cos (x+h)} = 2x - 3 \cdot 1 \cdot \sec^2 x \]

\[ f'(x) = 2x - 3 \sec^2 x \]

**Bonus:** SM 2.5.58

We have

\[ f(t) = \begin{cases} 
1 & 26 \leq t \leq 30 \\
g(t) & 30 \leq t \leq 34 \\
0 & 34 \leq t \leq 36 
\end{cases} \]

We want \( g(t) = a \cos (bt) + c \) so that \( f(t) \) is differentiable at \( t = 30 \) and \( t = 34 \).

This means

\[ \lim_{h \to 0^-} \frac{f(30+h) - f(30)}{h} = g'(30) \]

AND

\[ \lim_{h \to 0^+} \frac{f(34+h) - f(34)}{h} = g'(34) \]

AND

\[ g(30) = f(30) = 1, \quad g(34) = f(34) = 0 \]
\[ f(t) \text{ looks like:} \]

\[
\begin{align*}
\text{\frac{1}{2} a \text{ a period of cos} x \\
\text{a := amplitude} \\
0 & \quad 30 \\
& \quad 34 \\
\end{align*}
\]

From the graph (and properties of \(\cos x\)) we know

\[ a = \frac{\text{amplitude}}{2} \]

\[ c = \frac{\text{vertical shift}}{2} \]

We also know that half a period of \(\cos x\) should be equal to 4 (34-30). If \(f(t) = \cos(bt)\) the period is \(\frac{2\pi}{b}\).

So we want \(b = \frac{\pi}{4}\) \(\Rightarrow 8b = 2\pi \Rightarrow b = \frac{2\pi}{8} = \frac{\pi}{4}\)

But we also need a horizontal shift.

This leaves us with:

\[ g(t) = \frac{1}{2} \cos \left(\frac{\pi}{4}(t-30)\right) + \frac{1}{2} \]

Let's check \(g'(30)\) and \(g'(34)\):

\[ g'(t) = \frac{1}{2} \left(-\sin \left(\frac{\pi}{4}(t-30)\right)\right) + \frac{\pi}{4} + 0 \]

\[ = -\frac{\pi}{8} \sin \left(\frac{\pi}{4}(t-30)\right) \]

\[ g'(30) = -\frac{\pi}{8} \sin (0) = 0 \quad \text{and} \quad 0 = \lim_{h \to 0^-} \frac{f(30+h) - f(30)}{h} \]

\[ g'(34) = -\frac{\pi}{8} \sin (\pi) = 0 \quad \text{and} \quad 0 = \lim_{h \to 0^+} \frac{f(34+h) - f(34)}{h} \]

So \(g(t) = \frac{1}{2} \cos \left(\frac{\pi}{4}(t-30)\right) + \frac{1}{2}\) is the function that allows \(f(t)\) to be continuous and differentiable at \(t=30\) and \(t=34\), since \(g'(30) = 0 = \lim_{h \to 0^-} \frac{f(30+h) - f(30)}{h}\) and \(g'(34) = 0 = \lim_{h \to 0^+} \frac{f(34+h) - f(34)}{h}\).
Taylor's Theorem and Error in Tangent Line Approximations

Name:

1. In class we saw the following equation for the error in a tangent line approximation.

   \[ E(h) = \text{True Value} - \text{Approximate Value} = f(a + h) - (f(a) + f'(a)h). \]

   (a) Explain, using complete sentences, why we write \( E(h) \). Be sure to state what both \( E \) and \( h \) are. (This is related to what the function notation means.)

   (b) Explain, in complete sentences, how the approximation error changes as we get closer and farther from the value \( a \).

   (a) We write \( E(h) \) since the error is a function of \( h \), i.e. as \( h \) changes, so will the error.

   (b) The approximation error decreases as \( h \) decreases so that we are closer to \( a \); similarly as \( h \) increases and we are approximating values farther from \( a \), the approximation error increases.

2. For the following functions, find the equation of the tangent line at \( x = 0 \) and find a formula for the error \( E(h) \). Then approximate \( f(1) \) and evaluate the error in the approximation.

   (a) \( f(x) = \frac{1}{x+3} \)

   \[ f(0) = \frac{1}{3} \]

   \[ f'(x) = \frac{-1}{(x+3)^2} \]

   \[ f'(0) = -\frac{1}{9} \]

   \[ \text{tangent line at } x = 0: \]

   \[ y = \frac{1}{3} + -\frac{1}{9}(x-0) = \frac{1}{3} - \frac{1}{9}x \]

   \[ \text{tangent line: } y = f(a) + f'(a)h \]

   \[ f(1) \approx \frac{1}{3} - \frac{1}{9} = \frac{2}{9} \]

   \[ f(1) - f(0) = \frac{2}{9} - \frac{1}{9} = \frac{1}{36} \text{ as error} \]

   (b) \( f(t) = t^3 + 3t + 5 \)

   \[ f(0) = 5 \]

   \[ f'(t) = 3t + 3 \]

   \[ f'(0) = 3 \]

   \[ \text{tangent line at } x = 0: \]

   \[ y = 5 + 3(x-0) = 5 + 3x \]

   \[ f(1) \approx 5 + 3(1) = 8 \]

   \[ f(1) - f(0) = 9 - 8 = 1 \text{ as error} \]

   (c) \( f(x) = xe^x \)

   \[ f(0) = 0 \cdot e^0 = 0 \]

   \[ f(1) = 1 \cdot e' = e \]

   \[ f'(x) = xe^x + e^x \]

   \[ f'(0) = 0 e^0 + e^0 = 1 \]

   \[ \text{tangent line at } x = 0: \]

   \[ y = 0 + 1(x-0) = x \]

   \[ f(1) \approx 1 \]

   \[ f(1) - f(0) = e^1 - 1 \text{ as error} \]
Taylor’s Theorem and Error in Tangent Line Approximations

3. Pick one of the functions in (2.), graph the function and the tangent line and illustrate the error in your approximation to \( f(1) \).

The following figures show the graphs for each function in part (2.). They are only graphed over the interval \([-0.1, 1.1]\).
4. Pick one of the functions in (2.) along with its error formula \( E(h) \) and show that

\[
\lim_{h \to 0} E(h) = 0; \quad \text{and} \quad \lim_{h \to 0} \frac{E(h)}{h} = 0
\]

for (a)

\[
E(h) = f(a+h) - (f(a) + f'(a)h) \quad a=0
\]

\[
= \frac{1}{(0+h)+3} - \left( \frac{1}{a} - \frac{1}{q} h \right)
\]

\[
= \frac{1}{h+3} - \frac{1}{3} + \frac{1}{q} h \quad E(h) = \frac{1}{h+3} + \frac{h}{q} - \frac{1}{3}
\]

\[
\lim_{h \to 0} E(h) = \lim_{h \to 0} \frac{1}{h+3} + \frac{h}{q} - \frac{1}{3} = \frac{1}{3} + 0 - \frac{1}{3} = 0 \checkmark
\]

\[
\lim_{h \to 0} \frac{E(h)}{h} = \lim_{h \to 0} \frac{1}{h+3} + \frac{h}{q} - \frac{1}{3} \cdot \frac{(q(h+3))}{h} \frac{(q(h+3))}{(q(h+3))}
\]

\[
= \lim_{h \to 0} \frac{9 + h(h+3) - 3(h+3)}{h, 9(h+3)}
\]

\[
= \lim_{h \to 0} \frac{9 + h^2 + 3h - 3h - 9}{9h(h+3)}
\]

\[
= \lim_{h \to 0} \frac{h^2}{9h(h+3)} = \lim_{h \to 0} \frac{h}{9(h+3)} = \frac{0}{27} = 0 \checkmark
\]
For (b):

\[ E(h) = f(a + h) - (f(a) + f'(a) h) \quad a = 0 \]

\[ = (0 + h)^2 + 3(0 + h) + 5 - (5 + 3h) \]

\[ = h^2 + 3h + 5 - 5 - 3h = h^2 \quad E(h) = h^2 \]

\[ \lim_{h \to 0} E(h) = \lim_{h \to 0} h^2 = 0 \quad \checkmark \]

\[ \lim_{h \to 0} \frac{E(h)}{h} = \lim_{h \to 0} \frac{h^2}{h} = \lim_{h \to 0} h = 0 \quad \checkmark \]

for (c):

\[ E(h) = f(a + h) - (f(a) + f'(a) h) \quad a = 0 \]

\[ = (0 + h)e^{(0 + h)} - (0 + 1 \cdot h) \]

\[ = h e^h - h = h(e^h - 1) \quad E(h) = h(e^h - 1) \]

\[ \lim_{h \to 0} E(h) = \lim_{h \to 0} h(e^h - 1) = 0 (1-1) = 0 \quad \checkmark \]

\[ \lim_{h \to 0} \frac{E(h)}{h} = \lim_{h \to 0} \frac{h(e^h - 1)}{h} = \lim_{h \to 0} e^h - 1 = 1 - 1 = 0 \quad \checkmark \]