Problem 1. (20 points)

Indicate whether each of the following statements concerning the function

$$f(x) = |x|, \quad -\infty < x < +\infty,$$

is TRUE or FALSE. Then briefly explain the reason for your choice.

a) The function f is a linear function.

b) The function f is continuous at $x = 0$.

c) The function f is bounded on *some* open interval containing $x = 0$.

d) The function f is differentiable at $x = 0$.

Problem 2. (15 points)

Find the exact solution to the following initial value problem. For full credit, show your CHECK that your solution is correct.

$$y' = 2\,y, \qquad y(3) = 4.$$

Problem 3. (5 points)

TRUE or FALSE? (Briefly explain your answer.) "If $y' = F(y)$, then $y'' = F'(y)$."

Problem 4. (20 points)

The questions below concern the initial value problem:

$$y' = 2/y, \qquad y(0) = 5.$$

a) Does the Existence Theorem guarantee that this initial value problem has at least one solution? Explain.

b) Does the Uniqueness Theorem guarantee that this initial value problem has a *unique* solution? Explain.

Problem 5. (10 points)

Complete the following table to estimate $y(1)$, using Euler's Method with $\Delta t = 1/3$, where $y(t)$ is the solution of the initial value problem

$$y' = 2/y, \quad y(0) = 5.$$

t	$Y(t)$	ΔY
0		
1/3		
2/3		
1		

Problem 6. (10 points)

Euler's Method was applied to the IVP above with successively smaller stepsizes. Here are two successive estimates of $y(1)$:

$$Y_4 = 5.74486121 \quad \text{using stepsize } h$$
$$Y_5 = 5.74466212 \quad \text{using stepsize } h/3$$

Use this information and Richardson Extrapolation to find a better estimate of $y(1)$ (to the same number of decimal places as the estimates above). Show your work.

Problem 7. (20 points)

The following rate equation has unique solutions for all initial conditions:

$$y' = F(y) = (y-5)^3(y-1)$$

a) Find any equilibrium values. Show your work.

b) Classify the equilibrium values you found as asymptotically stable or not asymptotically stable.

c) Find any inflection values. Show your work.

d) Sketch a complete portrait of representative solutions to this rate equation. Be sure to show their concavity correctly.

Problem 8. (20 points)

Let $f(x) = e^{x^2}$.

a) Find the first-order Taylor polynomial $P_1(x)$ for f about $a = 1$.

b) Find the equation of the line tangent to the graph of f at the point $(1, f(1))$.

c) Find the second-order Taylor polynomial $P_2(x)$ for f about $a = 1$.

d) Should $P_1(1.03)$ or $P_2(1.03)$ be the better estimate of $f(1.03)$? Explain.

Problem 9. (20 points)

The graphs of $\ln x$ and $1/x$ intersect at one point. The x-coordinate of this point is the positive root of

$$g(x) = \ln(x) - 1/x.$$

a) Using your calculator, graph $\ln x$ and $1/x$ on the same plot, then [trace] the graphs to estimate the value of x where they intersect. **Sketch your plot and indicate this value below.**

b) Use your estimate above as a starting guess, then use Newton's Method (with your calculator) to improve the estimate. Continue until **ten** digits have stabilized. **Record your final answer below.**

c) Refer to the Convergence Theorem for Newton's Method. Determine $F(x)$ and $F'(x)$ for this problem. (You may find it convenient to simplify F before calculating F'.)

$F(x) =$

$F'(x) =$

d) **Explain** why the Convergence Theorem guarantees that Newton's Method, using the starting value you chose, does converge to the root you want. Include a sketch of the **graph of** F'.

Problem 10. (20 points)

These questions concern the function $f(x, y) = \sqrt{xy}$.

a) Find the partial derivatives $f_x(x, y)$ and $f_y(x, y)$.

b) Show that the point $(x_0, y_0, z_0) = (1, 1, 1)$ is on the graph of f.

c) Find the equation (in initial value form) of the plane tangent to the graph of f at the point $(x_0, y_0, z_0) = (1, 1, 1)$.

d) Find the slope of the line tangent to the level curve $f(x, y) = 1$ at $(x_0, y_0) = (1, 1)$.

Problem 11. (20 points)

The function $f(x, y) = \sqrt{xy}$ is called the *geometric mean* of x and y.
You are asked to find the point (x^*, y^*) within the triangular region,
defined by $x > 0$, $y > 0$, and $3x + 2y \leq 4$,
where f is maximized.

a) Refer to the contour plot below. Mark the approximate location of the point where the geometric mean is maximized, subject to the constraints above. Briefly explain your choice.

b) Use an appropriate method to find the exact coordinates of this point. Show your work.

Problem 12. (5 points)

A contour plot for a function f on the domain $-2 \leq x \leq 2$, $-2 \leq y \leq 2$ is shown below. It has two critical points, in this domain. **Put a "dot" on the approximate locations of these critical points, label them A and B, then indicate below whether they are local minima, local maxima, or saddle points.**

A:

B:

Problem 13. (15 points)

Evaluate the following limits, if they exist, showing any necessary work or explanations. If a limit does not exists, indicate that and give an explanation.

a) $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$

c) $\lim_{x \to 0} \frac{1}{x^3}$

b) (*Hint:* See Problem 7.)

Let $y(t)$ be the solution of the initial value problem $y' = (y-5)^3(y-1)$, $y(0) = 3$.

Then $\lim_{t \to \infty} y(t) =$

THEOREM: (**Taylor's Theorem**)

Suppose $f(x)$ is continuous on an open interval I containing a and $a+h$, and suppose $f'(a)$ exists. Then

$$f(a+h) = f(a) + f'(a)h + E_1(h),$$

where the approximation error $E_1(h)$ satisfies $\lim_{h\to 0} E_1(h) = 0, \quad \lim_{h\to 0} \frac{E_1(h)}{h} = 0.$

Moreover,

if $f'(x)$ also exists and is continuous on this same open interval I, and if $f''(a)$ exists, then

$$f(a+h) = f(a) + f'(a)h + \frac{1}{2}f''(a)h^2 + E_2(h),$$

where the approximation error $E_2(h)$ satisfies

$$\lim_{h\to 0} E_2(h) = 0, \quad \lim_{h\to 0} \frac{E_2(h)}{h} = 0, \quad \lim_{h\to 0} \frac{E_2(h)}{h^2} = 0.$$

THEOREM: (**Existence of Solutions to an IVP**)

Suppose the slope function $F(y)$ is *continuous* on an open interval containing the initial value y_0. Then for each initial time t_0, there is an open interval containing t_0 on which *at least one* solution to the following initial value problem exists:

$$y' = F(y), \qquad y(t_0) = y_0.$$

THEOREM: (**Uniqueness of Solutions to an IVP**)

Suppose there is an open interval, containing the intial value y_0, on which

i) $F'(y)$ exists, and
ii) $F'(y)$ is *bounded.*

Then for each initial time t_0, there is an open interval containing t_0 on which *exactly one* solution to the following initial value problem exists:

$$y' = F(y), \qquad y(t_0) = y_0.$$

THEOREM: (**Convergence of Newton's Method**)

Let I be an open interval with the following properties:

i) g has a unique root r in I,
ii) $g'(r) \neq 0$ (i.e. r is a simple root of g),
iii) g, g' and g'' are continuous on I,
iv) and $-2+\epsilon < F'(x) < 0-\epsilon$, for some $0<\epsilon<1$, where $F(x) = -g(x)/g'(x)$.

If we choose $x_0 \in I$, and let $X_0 = x_0$ and $X_{n+1} = X_n + F(X_n)$,

then $\lim_{n\to\infty} X_n = r.$ Moreover, $\lim_{n\to\infty} \frac{X_{n+1}-r}{(X_n-r)^2}$ exists and is finite.